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Log-concavity of the partition function

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Abstract We prove that the partition function $p(n)$ is log-concave for all $n > 25$. We then extend the results to resolve two related conjectures by Chen and one by Sun. The proofs are based on Lehmer's estimates on the remainders of the Hardy–Ramanujan and the Rademacher series for $p(n)$.

Keywords Integer partition · Partition function · Log-concave sequence · Asymptotic analysis · Error estimates

Mathematics Subject Classification 05A17 · 11N37 · 65G99

1 Introduction

A sequence $\{a_n\}$ is *log-concave* if it satisfies

$$a_n^2 - a_{n-1}\,a_{n+1} \geq 0 \quad \text{for all} \;\; n\,.$$

Notable examples of log-concave sequences are the binomial coefficients, the Stirling numbers, the Bessel numbers, etc. (see [5,19,22] for well-written surveys). Despite the interest in log-concave sequences and the detailed asymptotics of the partition function in combinatorics, the log-concavity of $p(n)$ for all $n > 25$ remained an open problem (see Sect. 7). Note that it fails for all odd values $n \leq 25$.

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Theorem 1.1 *Sequence $p(n)$ is log-concave for all $n > 25$.*

This intuition behind the theorem comes from explicit calculations for small n, and the Hardy–Ramanujan asymptotic formula [13]

$$p(n) \sim \frac{1}{4\sqrt{3}n} \, e^{\pi\sqrt{\frac{2}{3}n}} \quad \text{as } n \to \infty. \tag{1}$$

The function on the r.h.s. of (1) is easily shown to be log-concave, but without guaranteed error bounds there is no way of knowing precisely when the asymptotic formula dominates the calculation. Our proof is based on the estimates by Lehmer [17,18] which provide improved explicit guaranteed error estimates valid for all n. We use these to prove the theorem for all $n \geq 2{,}600$, and check log-concavity for smaller values using MATHEMATICA.

We turn to the following two recent conjectures by William Chen which partially motivated our study, see [7, pp. 117–121].

Conjecture 1.2 (Chen). *For all $n > 1$, we have*

$$\frac{p(n-1)}{p(n)}\left(1+\frac{1}{n}\right) > \frac{p(n)}{p(n+1)}. \tag{2}$$

Conjecture 1.3 (Chen). *For all $n > m > 1$, we have*

$$p(n)^2 - p(n-m)\,p(n+m) \geq 0. \tag{3}$$

Note that inequality (2) goes in the opposite direction to log-concavity. We fully establish Conjecture 1.2 in Theorem 4.1 and show that the $(1 + 1/n)$ term can be further sharpened to $(1+O(n^{-3/2}))$, see Theorem 4.2. The inequality (3) is sometimes called *strong log-concavity* when extended to all $m \geq 1$; we prove Conjecture 1.3 in Theorem 5.1.

The rest of the paper is structured as follows. We first prove Theorem 1.1 in Sect. 2. In a short but technical Sect. 3, we show that the proof of Theorem 1.1 gives sharp results on the asymptotic rate of decay of log-concavity. These are then used to prove Chen's conjectures (Sects. 4 and 5), and Sun's strongly related conjecture (Sect. 6). We conclude with final remarks in Sect. 7.

2 Proof of Theorem 1.1

The log-concavity is equivalent to

$$\log a_{n-1} - 2\log a_n + \log a_{n+1} \leq 0 \quad \text{for all } n,$$

which explains the name.

Recall the Rademacher's convergent series for $p(n)$, based on the Hardy–Ramanujan asymptotic series [20] (see also e.g. [2,12]). Let $\mu(n) = \frac{\pi}{6}\sqrt{24n-1}$, which we abbreviate to μ whenever convenient. We have

$$p(n) = \frac{\sqrt{12}}{24n-1} \sum_{k=1}^{N} A_k^*(n) \left[\left(1 - \frac{k}{\mu}\right) e^{\mu/k} + \left(1 + \frac{k}{\mu}\right) e^{-\mu/k} \right] + R_2(n, N), \tag{4}$$

where $A_k^*(n)$ is a complicated arithmetic function that will not be needed in its full generality for our analysis; see [17,18] for the complete definition. We shall only need that $A_1^*(n) = 1$ and $A_2^*(n) = (-1)^n/\sqrt{2}$ for all positive n. Recall also Lehmer's error bound [17]

$$|R_2(n, N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}} \left[\left(\frac{N}{\mu}\right)^3 \sinh \frac{\mu}{N} + \frac{1}{6} - \left(\frac{N}{\mu}\right)^2 \right], \tag{5}$$

valid for all positive n and N. Define for all $n \geq 1$

$$T(n) := \frac{\sqrt{12}}{24n-1} \left[\left(1 - \frac{1}{\mu}\right) e^{\mu} + \frac{(-1)^n}{\sqrt{2}} e^{\mu/2} \right]. \tag{6}$$

The function $T(n)$ contains the three largest individual contributions to the sum, and is a refinement of the right-hand side of Eq. (1). Define also the remainder term

$$R(n) := \frac{d}{\mu^2} \left[\left(1 + \frac{1}{\mu}\right) e^{-\mu} - \frac{(-1)^n}{\sqrt{2}} \frac{2}{\mu} e^{\mu/2} + \frac{(-1)^n}{\sqrt{2}} \left(1 + \frac{2}{\mu}\right) e^{-\mu/2} \right] + R_2(n, 2), \tag{7}$$

where $d = \frac{\pi^2}{6\sqrt{3}}$. We may rewrite Eq. (4) as

$$p(n) = T(n) + R(n). \tag{8}$$

Lemma 2.1 *Suppose $f(x)$ is a positive, increasing function with two continuous derivatives for all $x > 0$, and that $f'(x) > 0$ and decreasing, and $f''(x) < 0$ is increasing for all $x > 0$. Then*

$$f''(x-1) < f(x+1) - 2f(x) + f(x-1) < f''(x+1) \quad \textit{for all } x > 1.$$

Lemma 2.2 *Let*

$$T_1(n) := 2 \log T(n) - \log T(n-1) - \log T(n+1).$$

Then, for all $n \geq 50$, we have

$$\frac{24\pi}{(24(n+1)-1)^{3/2}} - \frac{3}{n^2} < T_1(n) < \frac{24\pi}{(24(n-1)-1)^{3/2}} + e^{-C\sqrt{n}/10}, \tag{9}$$

where $C = \pi\sqrt{\frac{2}{3}}$.

Proof We prove a stronger result, that for all $n \geq 2$, we have

$$\log T(n) = \log d + \mu + \log(\mu - 1) - 3\log\mu + \log\left(1 + \frac{(-1)^n e^{-\mu/2}\mu^3}{d\sqrt{2}(\mu - 1)}\right).$$

By Lemma 2.1,

$$T_1(n) > -\mu''(n+1) - (\log(\mu(n+1) - 1))'' + 3\log(\mu(n-1))'' + G(n), \quad (10)$$

where $G(n)$ is the expression with the more complicated logs. This simplifies to

$$T_1(n) > \frac{24\pi}{(24(n+1)-1)^{3/2}} + \frac{288\pi(-3+\pi\sqrt{24(n+1)-1})}{(24(n+1)-1)^{3/2}(-6+\pi\sqrt{24(n+1)-1})^2} - \frac{3 \times 288}{(24(n-1)-1)^2} + G(n),$$

valid for all $n \geq 2$. Let $x_n = \frac{(-1)^n e^{-\mu/2}\mu^3}{d\sqrt{2}(\mu-1)}$. Using $\log(1 - |x|) \leq -\log(1 - |x|)$ for all $|x| < 1$ and $\log(1 - x) \geq -x/(1 - x)$ for $0 < x < 1$, we have

$$G(n) > 4\log(1 - |x_{n+1}|) > \frac{-4|x_{n+1}|}{1 - |x_{n+1}|} > -e^{-C\sqrt{n}/10}, \quad \text{for all } n \geq 50.$$

For the upper bound, similarly note that

$$G(n) < -4\log(1 - |x_{n-1}|) < \frac{4|x_{n-1}|}{1 - |x_{n-1}|} < e^{-C\sqrt{n}/10}, \quad \text{for all } n \geq 50,$$

as desired. □

Lemma 2.3 *Let* $y_n = |R(n)|/T(n)$. *Then*

$$\log\left[\frac{(1 - y_n)(1 - y_n)}{(1 + y_{n-1})(1 + y_{n+1})}\right] > -e^{-C\sqrt{n}/10}, \quad \textit{for all } n \geq 10. \quad (11)$$

Proof Simply note that for all $n \geq 2$, we have

$$|R(n)| < 1 + \frac{16}{\mu^3} e^{\mu/2}, \quad (12)$$

and

$$0 < \frac{|R(n)|}{T(n)} < e^{-C\sqrt{n}/10}. \quad (13)$$

The result easily follows. □

Proposition 2.4 *Let* $p_2(n) = 2\log p(n) - \log p(n-1) - \log p(n+1)$. *Then, for all* $n \geq 2{,}600$, *we have*

$$\frac{1}{(24n)^{3/2}} < p_2(n) < \frac{2}{n^{3/2}}. \tag{14}$$

Proof We start by writing a double strict inequality for $p(n)$,

$$T(n)\left(1 - \frac{|R(n)|}{T(n)}\right) < p(n) < T(n)\left(1 + \frac{|R(n)|}{T(n)}\right), \quad \text{for all } n \geq 1.$$

We then take logs and apply the Lemmas, and obtain

$$\begin{aligned} &\frac{24\pi}{(24(n+1)-1)^{3/2}} - \frac{3}{n^2} - 2e^{-C\sqrt{n}/10} \\ &\quad < p_2(n) < \frac{24\pi}{(24(n-1)-1)^{3/2}} + 2e^{-C\sqrt{n}/10}. \end{aligned} \tag{15}$$

Now the result easily follows by a direct calculation. □

3 Rate of decay

The rate $O(n^{3/2})$ in Lemma 2.2 is asymptotically sharp. Based on the asymptotics of $T(n)$, we have the following corollary.

Corollary 3.1 *Let* $D(n) = 2\log p(n) - \log p(n-1) - \log p(n+1)$. *We have*

$$\lim_{n\to\infty} D(n)\,\frac{4n^{3/2}}{C} = 1, \quad \textit{where } C = \pi\sqrt{\frac{2}{3}}.$$

A plot of exact values for n up to 2,000 is given in Fig. 1. This asymptotic rate comes from the Taylor Series of $\mu \sim 2c\sqrt{n}$ starting with the second derivative term.

A refinement of the rate in Theorem 3.1 to higher order terms follows easily as long as we also include the higher order terms of $-2\log(\mu)$. Since $(2\log(\mu))'' \sim O(n^{-2})$, these terms did not appear in the first-order asymptotic expansion, but higher order expansions must take them into account.

Let

$$h_1(x) = \frac{4}{Cx^{3/2}} \quad \text{and} \quad h_2(x) = \frac{-288}{(-1+24x)^2}.$$

Fix an integer $k \geq 0$. Denote by $L_k^+(n)$ the first k even terms in the Taylor series expansion of $h_1(n+1) + h_2(n+1)$ evaluated at $x_0 = n$, $k = 0, 1, \ldots$. Similarly, denote by $L_k^-(n)$ the first k even terms in the Taylor series expansion of $h_1(n+1)$ evaluated at $x_0 = n$, plus the first $k-1$ even terms in the Taylor series expansion of $h_2(n+1)$ evaluated at $x_0 = n$, $k = 1, 2, \ldots$, with $L_0^-(n) = h_1(n)$.

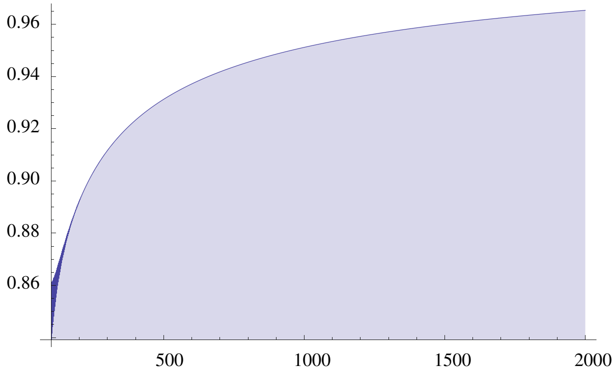

Fig. 1 A plot of $n^{\frac{3}{2}} D(n)/(\frac{\pi}{\sqrt{24}})$, for $n = 1, \ldots, 2000$

Corollary 3.2 *For every integer $k \geq 0$, we have*

$$D(n) \sim L_k^+(n), \quad D(n) \sim L_k^-(n) \quad \text{as } n \to \infty.$$

4 Reverse inequality

In this section, we establish and further strengthen Conjecture 1.2.

Theorem 4.1 *(formerly Chen's Conjecture 1.2) For all $n > 1$, we have*

$$\frac{p(n-1)}{p(n)}\left(1+\frac{1}{n}\right) > \frac{p(n)}{p(n+1)}. \tag{16}$$

Proof We can rewrite Eq. (16) as

$$p_2(n) < \log\left(1+\frac{1}{n}\right).$$

By Proposition 2.4, for all $n \geq 2{,}600$ we have

$$p_2(n) < \frac{2}{n^{3/2}} < \frac{1}{n+1} < \log\left(1+\frac{1}{n}\right).$$

We then check numerically in MATHEMATICA that Eq. (17) holds for all positive $n < 2{,}600$. □

We next strengthen Theorem 4.1 to the correct rate.

Theorem 4.2 *For all $n > 6$, we have*

$$\frac{p(n-1)}{p(n)}\left(1+\frac{240}{(24n)^{3/2}}\right) > \frac{p(n)}{p(n+1)}. \tag{17}$$

Proof The theorem follows similarly from the proof above, since for all $n \geq 22$, we have

$$p_2(n) < \frac{2}{n^{3/2}} < \frac{240}{(24n)^{3/2} + 240} < \log\left(1 + \frac{240}{(24n)^{3/2}}\right).$$

□

Conjecture 4.3 *For all $n \geq 45$, we have*

$$\frac{p(n-1)}{p(n)}\left(1 + \frac{\pi}{\sqrt{24}\, n^{3/2}}\right) > \frac{p(n)}{p(n+1)}.$$

We have checked this conjecture in MATHEMATICA for values of $n \leq 8{,}000$, with violations occurring every even number less than 45. Lemma 2.2 comes close to a proof, but one would have to check carefully the error involved in replacing the term $24(n-1) - 1$ with $24n$, as well as the exponential contribution.[1]

5 Strong log-concavity

In this section, we establish Conjecture 1.3 on strong log-concavity of the partition function.

Theorem 5.1 *(formerly Chen's Conjecture 1.3) For all $n > m > 1$, we have*

$$p(n)^2 - p(n-m)\, p(n+m) > 0. \tag{18}$$

Proof First, recall that log-concavity implies strong log-concavity (see e.g. [21])

$$a_k\, a_\ell \;\leq\; a_{k+i}\, a_{\ell-i}\,,$$

for all $0 \leq k \leq \ell \leq n$ and $0 \leq i \leq k - \ell$. Take $k = n - m$, $\ell = n + m$, and $i = m$, to obtain

$$p(n)^2 - p(n-m)\, p(n+m) > 0 \quad \text{for all} \;\; n > 25 + m\,. \tag{19}$$

To resolve the remaining cases, we use the following classical bounds which hold for all $m \geq 1$:

$$e^{C\sqrt{m}} > p(m) > \frac{e^{2\sqrt{m}}}{2\pi\, m\, e^{1/6m}}.$$

The first inequality was given in [9], and the second is from Sect. 2 of [13]. Next, observe that

$$p(n-m) < p(25) < 2{,}000 \quad \text{for all}\; 1 \leq n - m \leq 25\,.$$

[1] Conjecture 4.3 was proved by Chen et al. [8].

Hence, for all $1 \le n - m \le 25$, we have

$$p(n)^2 \ \ge \ p(m+1)^2 \ \ge \ p(25)\, p(25+m) \ \ge \ p(n-m)\, p(n+m),$$

and thus it suffices to prove the middle inequality for all $m \ge 2$. But this is equivalent to

$$2\log p(m+1) - \log p(25) - \log p(25+m) > 0,$$

and by the inequalities above it is sufficient to establish for which values of m the following hold:

$$4\sqrt{m+1} - 2\log(m+1) - \frac{1}{3(m+1)} - 2\log(2\pi) - \log 2{,}000 - C\sqrt{m+25} > 0.$$

This is easily seen to hold for $m \ge 300$, and a calculation in MATHEMATICA easily establishes the remaining cases. □

6 Sun's conjecture

In [24], a similar conjecture is stated for $q(n) := p(n)/n$. We prove it here.

Theorem 6.1 *(formerly Sun's Conjecture). The sequence $\{q(n)\}_{n \ge 31}$ is log-concave.*

Proof Let $q_2(n) = 2\log q(n) - \log q(n-1) - \log q(n+1)$. Then

$$\log q_2(n) = \log p_2(n) - \log\left(\frac{n^2}{(n-1)(n+1)}\right).$$

By Lemma 2.1 with $f(x) = \log x$, we have

$$\frac{1}{(n+1)^2} < \log\left(\frac{n^2}{(n-1)(n+1)}\right) < \frac{1}{(n-1)^2}.$$

We simply need to modify Eq. 15 to include this new error, i.e.,

$$\begin{aligned} &\frac{24\pi}{(24(n+1)-1)^{3/2}} - \frac{3}{n^2} - 2e^{-C\sqrt{n}/10} - \frac{1}{(n+1)^2} \\ &\quad < q_2(n) < \frac{24\pi}{(24(n-1)-1)^{3/2}} + 2e^{-C\sqrt{n}/10}. \end{aligned} \qquad (20)$$

The direct calculation is the same as in the proof of Lemma 2.4, which establishes the log-concavity of $q(n)$ for $n \ge 2{,}600$. Since Sun checked numerically the log-concavity of $q(n)$ for $n \le 10^5$, see [24], this completes the proof. □

7 Final remarks

7.1 The log-concavity conjecture

Let us mention that although the conjecture behind Theorem 1.1 seems to be folklore, we were unable to locate it in the literature. The problem was revived by Chen [7], who checked it for all $n \le 8{,}000$, by Janoski (see the discussion below in Sect. 7.3), and most recently by Burde on *MathOverflow*.[2] After this paper was finished, we learned that it was also independently conjectured by Andrews and Hirschhorn.[3] Most recently, a different but somewhat related investigation was made in [3].

7.2 Historical background

The classical Hardy–Ramanujan formula is

$$p(n) = \frac{\sqrt{12}}{24n-1}\sum_{k=1}^{N} A_k^*(n)\left[\left(1-\frac{k}{\mu}\right)e^{\mu/k}\right] + R_1(n,N). \tag{21}$$

They derived the explicit forms of the $A_k^*(n)$ for $k = 1, \ldots, 18$ and all n as a linear combination of cosines [13]. This allowed them to compute expansions for various small values of n that were checked by MacMahon. In particular, two terms of their series give

$$p(n) \sim \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{C\lambda_n}}{\lambda_n}\right) + \frac{(-1)^n}{2\pi}\frac{d}{dn}\left(\frac{e^{\frac{1}{2}C\lambda_n}}{\lambda_n}\right) + O(e^{D\sqrt{n}}), \tag{22}$$

where

$$C = \pi\sqrt{\frac{2}{3}}, \qquad \lambda_n = \sqrt{24n-1} \quad \text{and any} \quad D > \frac{C}{3}.$$

This series was also extended to include $N = O(\sqrt{n})$ terms with an error of $O(n^{-1/4})$. However, it was later shown by Lehmer [16] that this series diverges if extended to infinity. The first term of the series easily suffices to establish log-concavity for sufficiently large n (see Sect. 7.2), but there is no natural way to make these bounds explicit. Rademacher modified the formula and produced a convergent series in [20]. Most recently, this series was used by Janoski [14, Sect. 1.2], to claim to prove Theorem 1. Unfortunately, this proof has a serious logical flaw (see below).

This relation has since been interpreted in the popular literature as

$$p(n) = \frac{e^{C\sqrt{n}}}{4\sqrt{3}\,n}\left(1 + O(n^{-1/2})\right).$$

[2] See http://tinyurl.com/kkc6fwf.

[3] Personal communication.

It follows then, that one cannot base any heuristic interpretation of log-concavity on this simplest form, as it has an error that is larger than the asymptotic rate of decay given by Theorem 3.1.

One can, however, embrace the *entire* first-order term of the Hardy-Ramanujan series, and obtain

$$p(n) = \frac{e^{C\sqrt{n-\frac{1}{24}}}}{4\sqrt{3}\,\left(n-\frac{1}{24}\right)}\left(1-\frac{1}{C\sqrt{n-\frac{1}{24}}}\right)\left(1+O(e^{-\mu/2})\right).$$

The relative error term above decreases exponentially, and hence it easily follows that $\{p(n)\}$ is *asymptotically log-concave*, i.e., that $\{p(n)\}$ is log-concave for all $n > n_0$, for some *unspecifiable* n_0.[4] Without a concrete error bound in the form of a strict inequality, one cannot in general obtain the result.

The *Rademacher series* for $p(n)$ is a convergent series of the form

$$p(n) = \frac{\sqrt{12}}{24n-1}\sum_{k=1}^{\infty} A_k^*(n)\left[\left(1-\frac{k}{\mu}\right)e^{\mu/k} + \left(1+\frac{k}{\mu}\right)e^{-\mu/k}\right].$$

This series was shown to converge by Rademacher. Equation (4) is the truncation of this series to N terms, and Rademacher provided a *strict* inequality on the absolute value of the error term $R_2(n, N)$ for all n and N. This was later improved by Lehmer, and we have used his estimates even though in principle Rademacher's estimates would have been sufficient.

An important part of the proof of Theorem 1.1 is the form of the error. Rademacher's upper bound can be written for some positive constant c' as

$$|R_2(n, N)| < c' e^{\mu/N}\frac{\sqrt{N}}{\mu} \quad \text{for all } n,\ N \geq 1.$$

Lehmer's upper bound can be written for some other positive constant c'' as

$$|R_2(n, N)| < c'' e^{\mu/N}\frac{N^3}{\mu^3} \quad \text{for all } n,\ N \geq 1.$$

Thus, neither Rademacher's nor Lehmer's upper bounds for $N = 1$ provide a sharp enough bound to have a relative error $o(n^{-3/2})$. In particular, the exponent μ/N implies that we must take $N \geq 2$ in order to obtain an exponentially decaying error term relative to the first term. This demonstrates that one cannot obtain even a proof of asymptotic log-concavity using this error analysis with $N = 1$.

[4] See e.g., Speyer's calculation in http://tinyurl.com/nyq2zrn.

7.3 Janoski's thesis

Recently, Janoski in her thesis [14] claimed to have a complete proof of log-concavity, for all $n > 25$. Unfortunately, there is a serious technical error early in the proof, invalidating the argument. Specifically, Janoski writes on page 9 the following inequality:

$$\frac{1}{\pi\sqrt{2}} \sum_{k=3}^{C\sqrt{t}} A_k(n)\sqrt{k}\, d(n,k) > C\sqrt{t}\, \frac{\sqrt{3}}{\pi\sqrt{2}}\, A_3(n)\, d(n,3),$$

where

$$t = n - \frac{1}{24}, \qquad A_k(n) = A_k^*(n)/\sqrt{k}, \quad \text{and} \quad d(n,k) = \left(\frac{\sinh C\sqrt{t}/k}{\sqrt{t}}\right)'.$$

It seems, the author assumes that the functions $A_k(n)$ are positive and monotonic in k. We have checked this inequality for various small values of n, and have found several counterexamples, such as n equal to 27, 36, 87, or 744. Furthermore, the calculation bounding the absolute error $R_2(n, N)$ is incorrect. The rest of the analysis in [14] is based on bounding from below this faux lower bound term, which has a relative error $O(e^{-2\mu/3})$ (though not in the form of a strict inequality!), and so naturally the numerical results would appear to confirm the entire result.

7.4 Computer calculations

All of the calculations for small values of n mentioned in the proof above are trivial, if tedious. The calculations were done with MATHEMATICA, which has a built-in partition function. Let us note, however, that strictly speaking, the use of computer is not necessary to follow and verify the proofs of the Lemmas, as there are comprehensive tables of the partition function, see e.g., [10].

7.5 Combinatorial proof

It is natural to ask for a direct combinatorial proof of Theorem 1.1, e.g., in the style of proofs in [4,11,15,21]. Unfortunately, the previous attempts and the fact that log-concavity fails for small n, are very discouraging. It seems, even the simplest results for the partition function are difficult to prove directly (cf. [1]).

Note, however, that for related sequences such as *convex compositions* (or *stacks*), the asymptotic formulas are similar to (1), and thus not sharp enough for the analysis as in this paper (see [6,25]). A combinatorial proof is the best current hope for proving asymptotic log-concavity of such sequences.

7.6 Another Chen's conjecture

Note that Theorem 5.1 says that the sequences $\{p(mn + a)\}$ are log-concave, for all fixed $m > a \geq 0$. In [7], Chen also makes a related conjecture, that for all $a > b$ we have

$$p(an)^2 - p(an - bn)\, p(an + bn) > 0 \quad \text{whenever}\; an,\; bn \in \mathbb{N}.$$

Of course, this is a weak version of Conjecture 1.3. Now that Theorem 5.1 is proved, it also follows.

7.7 Another Sun's conjecture

Let $r(n) := \sqrt[n]{p(n)/n}$. In [24] (see also [23]), Sun conjectures that $\{r(n)\}$ is log-concave for $n \geq 60$. Unfortunately, the calculation for $r(n)$ does not follow immediately from our lemmas, since there is no simple lemma analogous to Lemma 2.1. However, it is easy to see that such result only requires a more careful bounding of elementary functions; we hope the reader will continue this investigation. Let us mention also that in the same paper, Sun makes similar conjectures for partitions into distinct part, supported by numerical evidence.

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