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# $r$-log-concavity of partition functions

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**Abstract** Let $\hat{\mathscr{L}}$ be the operator given by $\hat{\mathscr{L}}\{a_n\}_{n\geq 0} = \{a_{n+1}^2 - a_n a_{n+2}\}_{n\geq 0}$. A sequence $\{a_n\}_{n\geq 0}$ is called asymptotically $r$-log-concave if $\hat{\mathscr{L}}^k\{a_n\}_{n\geq N}$ are non-negative sequences for $1 \leq k \leq r$ and some integer $N$. Let $p(n)$ be the number of integer partitions of $n$. We prove that the sequence $\{p(n)\}_{n\geq 1}$ is asymptotically $r$-log-concave for any positive integer $r$. Moreover, we give a method to compute the explicit $N$ such that $\{p(n)\}_{n\geq N}$ is $r$-log-concave.

**Keywords** $r$-log-concavity · Partition function · Hardy–Ramanujan–Rademacher formula

**Mathematics Subject Classification** 05A17 · 11N37 · 65G99

## 1 Introduction

Let $p(n)$ denote the number of integer partitions of $n$, i.e., the number of ways of writing $n$ as the sum of positive integers where the order is irrelevant. Among the various interesting combinatorial properties, the following inequalities were conjectured by Chen [1] and solved by DeSalvo and Pak [5].

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**Proposition 1.1** *For all $n \geq 1$, we have*

$$\frac{p(n-1)}{p(n)}\left(1+\frac{1}{n}\right) > \frac{p(n)}{p(n+1)}.$$

**Proposition 1.2** *For all $n > m > 1$, we have*

$$p(n)^2 - p(n-m)p(n+m) \geq 0.$$

In the same paper, DeSalvo and Pak also proved Sun's conjecture [11] of the log-concavity of $\{q(n)\}_{n\geq 31}$, where $q(n)$ is the number of ways writing $n$ as the sum of distinct positive integers. Based on Lehmer's error bound and modified Bessel function, Chen et al. [4] proved the positivity of $(-1)^{r-1}\Delta \log p(n)$ for $r \geq 1$ and $n$ sufficiently large, where $\Delta$ is the difference operator. Chen and Zheng [3] improved DeSalvo and Pak's bound for $-\Delta^2 \log p(n-1)$ and proved the log-convexity of $\{\sqrt[n]{p(n)}\}_{n\geq 27}$ and $\{\sqrt[n]{p(n)/n}\}_{n\geq 61}$.

In this paper, we focus on the asymptotic $r$-log-concavity of $\{p(n)\}$. A sequence $\{a_n\}_{n\geq 0}$ is said to be asymptotically *$r$-log-concave* if there exists $N$ such that

$$\hat{\mathscr{L}}\{a_n\}_{n\geq N},\ \hat{\mathscr{L}}^2\{a_n\}_{n\geq N},\ \ldots,\ \hat{\mathscr{L}}^r\{a_n\}_{n\geq N} \tag{1.1}$$

are all non-negative sequences, where

$$\hat{\mathscr{L}}\{a_n\}_{n\geq 0} = \{a_{n+1}^2 - a_n a_{n+2}\}_{n\geq 0} \quad \text{and} \quad \hat{\mathscr{L}}^k\{a_n\}_{n\geq 0} = \hat{\mathscr{L}}\left(\hat{\mathscr{L}}^{k-1}\{a_n\}_{n\geq 0}\right).$$

Chen and Xia [2] gave a criterion for the 2-log-convexity of a sequence. We present a method of proving the asymptotic $r$-log-convexity and the asymptotic $r$-log-concavity of a sequence in [7]. More precisely, we have the following criterion for the asymptotic $r$-log-concavity.

**Theorem 1.3** *Let $\{a_n\}_{n\geq 0}$ be a positive sequence such that $\mathscr{R}^2 a_n = a_n a_{n+2}/a_{n+1}^2$ has the following asymptotic expression*

$$\mathscr{R}^2 a_n = 1 + \frac{c}{n^\alpha} + \cdots + o\left(\frac{1}{n^\beta}\right), \quad n \to \infty, \tag{1.2}$$

*where $0 < \alpha \leq \beta$. If $c < 0$ and $\alpha < 2$, then $\{a_n\}_{n\geq 0}$ is asymptotically $\lfloor \beta/\alpha \rfloor$-log-concave.*

To apply the criterion to $p(n)$, we firstly utilize the Hardy–Ramanujan–Rademacher formula [10] and the error estimation given by Lehmer [8] which are given below (2.1) (2.2) to derive an estimation for $p(n)$: for any $m$ there is an $n$ large enough such that

$$\left|\frac{p(n)}{T(n)} - 1\right| < 2^{m+1}\mu(n)^{-m},$$

where

$$T(n) = \frac{d}{\mu(n)^2}\left(1 - \frac{1}{\mu(n)}\right)e^{\mu(n)}$$

and $d = \frac{\pi^2}{6\sqrt{3}}$, $\mu(n) = \frac{\pi}{6}\sqrt{24n-1}$. Then by considering the Taylor expansion in terms of $\mu(n)$, we give an algorithm to establish the upper and lower bounds of $T(n+1)/T(n)$ up to an arbitrary accuracy. Finally, combining these bounds and the criterion for asymptotic $r$-log-concavity, we are able to find an explicit $N$ such that $\{p(n)\}_{n \geq N}$ is $r$-log-concave.

## 2 Error estimation of $p(n)$

In this section, we utilize the Hardy–Ramanujan–Rademacher formula and the error estimation given by Lehmer to derive an estimation for $p(n)$.

For any positive integers $n$ and $N$, the Hardy–Ramanujan–Rademacher formula reads

$$p(n) = \frac{d}{\mu(n)^2}\sum_{k=1}^{N}\frac{A_k(n)}{\sqrt{k}}\left[\left(1 - \frac{k}{\mu(n)}\right)e^{\mu(n)/k} + \left(1 + \frac{k}{\mu(n)}\right)e^{-\mu(n)/k}\right] + R_2(n, N), \tag{2.1}$$

where

$$d = \frac{\pi^2}{6\sqrt{3}}, \quad \mu(n) = \frac{\pi}{6}\sqrt{24n-1},$$

and $R_2(n, N)$ is the remainder, see Hardy and Ramanujan [6], Rademacher [10]. Lehmer [8,9] gave an upper bound for $R_2(n, N)$:

$$|R_2(n, N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}}\left[\left(\frac{N}{\mu(n)}\right)^3 \sinh\frac{\mu(n)}{N} + \frac{1}{6} - \left(\frac{N}{\mu(n)}\right)^2\right]. \tag{2.2}$$

It is known that $A_1(n) = 1$ and $A_2(n) = (-1)^n$ for all positive $n$. Therefore, by setting $N = 2$, we derive that

$$p(n) = T(n) + R(n),$$

where

$$T(n) = \frac{d}{\mu(n)^2}\left(1 - \frac{1}{\mu(n)}\right)e^{\mu(n)} \tag{2.3}$$

is the main term and $R(n)$ is the remainder term. By dropping off the terms with negative sign in (2.2), we derive that

$$|R_2(n,2)| < \frac{\pi^2 2^{-2/3}}{2\sqrt{3}} \left( \left( \frac{2}{\mu(n)} \right)^3 e^{\mu(n)/2} + \frac{1}{3} \right).$$

Therefore,

$$\left| \frac{R(n)}{T(n)} \right| < e^{-\mu(n)/2} \big( T_{1,2}(n) + T_{2,1}(n) + T_{2,2}(n) + R_{2,1}(n) + R_{2,2}(n) \big),$$

where

$$T_{1,2}(n) = \frac{\mu(n)+1}{\mu(n)-1} e^{-3\mu(n)/2},$$
$$T_{2,1}(n) = \frac{1}{\sqrt{2}} \frac{\mu(n)-2}{\mu(n)-1}, \quad T_{2,2}(n) = \frac{1}{\sqrt{2}} \frac{\mu(n)+2}{\mu(n)-1} e^{-\mu(n)},$$
$$R_{2,1}(n) = 12 \cdot 2^{1/3} \frac{1}{\mu(n)-1}, \quad R_{2,2}(n) = \frac{1}{2^{2/3}} \frac{\mu(n)^3}{\mu(n)-1} e^{-\mu(n)/2}.$$

It is straightforward to see that $T_{1,2}(n)$, $T_{2,2}(n)$, and $R_{2,1}(n)$ are decreasing functions of $n$. For $R_{2,2}(n)$, one may check that $\frac{x^3}{x-1} e^{-x/2}$ is decreasing for $x > 3$ and thus $R_{2,2}(n)$ is decreasing for $n \geq 2$ (in fact, for $n \geq 1$). $T_{2,1}(n)$ is increasing and its limit is $1/\sqrt{2}$ when $n$ tends to infinity. The other four terms tend to 0 when $n$ tends to infinity. One can compute that

$$T_{1,2}(30) + \frac{1}{\sqrt{2}} + T_{2,2}(30) + R_{2,1}(30) + R_{2,2}(30) \approx 1.986 < 2$$

and

$$T_{1,2}(421) + \frac{1}{\sqrt{2}} + T_{2,2}(421) + R_{2,1}(421) + R_{2,2}(421) \approx 0.99995 < 1.$$

Therefore,

$$\left| \frac{R(n)}{T(n)} \right| < 2e^{-\mu(n)/2} \quad \forall n \geq 30, \tag{2.4}$$

and

$$\left| \frac{R(n)}{T(n)} \right| < e^{-\mu(n)/2} \quad \forall n \geq 421. \tag{2.5}$$

Noting that $\lim_{n\to\infty} \mu(n) = +\infty$, for any given $m$, the error can be bounded by $(\mu(n)/2)^{-m}$ for $n$ sufficiently large.

**Lemma 2.1** *For any integer $m \geq 1$, there exists a real number*

$$N \leq \max\{1, 2m \log m\}$$

*such that*

$$x^m e^{-x} < 1 \quad \forall x \geq N.$$

*Proof* We see that $x^m e^{-x}$ is decreasing for $x > m$ by taking the derivative with respect to $x$.

If $m = 1$ or 2, we have $m^m/e^m < 1$ and we may take $N = m$.

If $m \geq 3$, we have

$$(2m \log m)^m e^{-2m \log m} = e^{m(\log(2 \log m/m))}.$$

Once again by taking the derivative with respect to $m$, we see that $2 \log m < m$ and hence

$$e^{m(\log(2 \log m/m))} < 1,$$

completing the proof. □

Given $m$, the integer $N$ such that

$$e^{-\mu(n)/2} \leq (\mu(n)/2)^{-m} \quad \forall n \geq N$$

can be computed by finding the first $N$ such that

$$\frac{\mu(N)/2}{\log(\mu(N)/2)} \geq m.$$

We give Table 1 of $N$ for $m = 1, 2, \ldots, 10$ as follows.

Finally, we consider $\mu(n)^{-m}$. We have

$$\mu(n)^{-m} = \left(\sqrt{\frac{2}{3}}\pi\right)^{-m} \left(n - \frac{1}{24}\right)^{-m/2}.$$

Since

$$\left(\frac{n}{n - \frac{1}{24}}\right)^{m/2}$$

**Table 1** The integer $N$ such that $e^{-\mu(n)/2} < 2^m \mu(n)^{-m}$ for $n \geq N$

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | 1 | 3 | 13 | 46 | 99 | 176 | 281 | 414 | 580 | 778 |

is a decreasing function of $n$, we derive that

$$\mu(n)^{-m} < \left(\pi\sqrt{\frac{2}{3}}\right)^{-m}\left(\frac{N}{N-\frac{1}{24}}\right)^{m/2} \cdot n^{-m/2} \quad \forall n \geq N. \tag{2.6}$$

In summary, we obtain the following estimation on the ratio $R(n)/T(n)$.

**Theorem 2.2** *Let $p(n)$ be the number of integer partitions of $n$ and $T(n)$ is given by (2.3). Then for any integer $m \geq 2$, there exists an integer $N$ with*

$$N \leq \max\left\{30,\ 2.56 \cdot m^2 \log^2 m\right\}$$

*such that*

$$|p(n)/T(n) - 1| < 2^{m+1}\mu(n)^{-m} \quad \forall n \geq N.$$

*Proof* Let $R(n) = p(n) - T(n)$. We have seen that for $n \geq 30$,

$$|R(n)/T(n)| < 2e^{-\mu(n)/2}$$

and there exists $N_0 \leq \max\{1,\ 4m\log m\}$ such that

$$e^{-\mu(n)/2} < 2^m\mu(n)^{-m} \quad \forall \mu(n) \geq N_0.$$

By (2.6), we have

$$n^{1/2} < 0.4\mu(n) \quad \forall n \geq 30.$$

Therefore, when

$$n \geq \max\{30,\ 2.56 \cdot m^2\log^2 m\},$$

we have

$$\mu(n) > \max\{1,\ 4m\log m\},$$

and hence $e^{-\mu(n)/2} < 2^m\mu(n)^{-m}$. □

## 3 Bounds for the ratio $p(n+1)/p(n)$

In this section, we show how to derive lower and upper bounds for $T(n+1)/T(n)$ so that we obtain an estimation of $p(n+1)/p(n)$.

Recall that

$$T(n) = \frac{\pi^2}{6\sqrt{3}}\frac{\mu(n)-1}{\mu(n)^3}e^{\mu(n)}.$$

Instead of considering the Taylor expansion in $n$, we consider the Taylor expansion in $\mu(n)$. For brevity, we denote $\mu(n)$ and $\mu(n+1)$ by $\mu$ and $\mu_+$, respectively.

Noting that

$$n = \frac{3}{2\pi^2}\mu^2 + \frac{1}{24},$$

we have

$$\mu_+ = \mu\left(1 + \frac{2\pi^2}{3\mu^2}\right)^{1/2}.$$

For any integer $m$, let $m' = \lfloor m/2 \rfloor$. By Taylor's Theorem, we have

$$\begin{aligned}\left(1 + \frac{2\pi^2}{3\mu^2}\right)^{1/2} &= \sum_{k=0}^{m'} \binom{1/2}{k} \left(\frac{2\pi^2}{3}\right)^k \mu^{-2k} \\ &\quad + \binom{1/2}{m'+1} \left(\frac{2\pi^2}{3\mu^2}\right)^{m'+1} (1+\xi)^{\frac{1}{2}-m'-1},\end{aligned}$$

where $0 < \xi < \frac{2\pi^2}{3\mu^2}$. Denote

$$\mu_1 = \sum_{k=0}^{m'} \binom{1/2}{k} \left(\frac{2\pi^2}{3}\right)^k \mu^{-2k}.$$

and

$$\varepsilon_1 = \left|\binom{1/2}{m'+1}\right| \left(\frac{2\pi^2}{3}\right)^{m'+1} \mu^{-2m'-2}.$$

We have

$$\mu_1 - \varepsilon_1 < \frac{\mu_+}{\mu} < \mu_1 + \varepsilon_1. \tag{3.1}$$

Now we consider the ratio

$$\frac{T(n+1)}{T(n)} = \frac{\mu_+ - 1}{\mu - 1} \cdot \frac{\mu^3}{\mu_+^3} \cdot e^{\mu_+ - \mu}$$

term by term.

For the first factor, we have

$$\frac{\mu_+ - 1}{\mu - 1} = \frac{\frac{\mu_+}{\mu} - \frac{1}{\mu}}{1 - \frac{1}{\mu}}.$$

Since

$$\sum_{k=0}^{m} \mu^{-k} < \left(1 - \frac{1}{\mu}\right)^{-1} < \sum_{k=0}^{m} \mu^{-k} + 2\mu^{-m-1},$$

we have

$$\left(\mu_1 - \varepsilon_1 - \frac{1}{\mu}\right) \sum_{k=0}^{m} \mu^{-k} < \frac{\mu_+ - 1}{\mu - 1} < \left(\mu_1 + \varepsilon_1 - \frac{1}{\mu}\right) \left(\sum_{k=0}^{m} \mu^{-k} + 2\mu^{-m-1}\right).$$

Expanding the left-hand side of the above inequality, we obtain a polynomial in $\mu^{-1}$:

$$\sum_{k=0}^{m} c_k \mu^{-k} + \sum_{k=m+1}^{l} c_k \mu^{-k}.$$

Let

$$\tilde{c}_k = \begin{cases} -c_k & \text{if } c_k < 0, \\ 0 & \text{if } c_k \geq 0. \end{cases}$$

We then have

$$\frac{\mu_+ - 1}{\mu - 1} > \sum_{k=0}^{m} c_k \mu^{-k} - \sum_{k=m+1}^{l} \tilde{c}_k \mu^{-k}.$$

Noting further that $\mu \geq \mu_0 = \mu(n_0)$ for $n \geq n_0$, we thus derive a lower bound for $(\mu_+ - 1)/(\mu - 1)$:

$$\sum_{k=0}^{m} c_k \mu^{-k} - \mu^{-m-1} \sum_{k=m+1}^{l} \tilde{c}_k \mu_0^{-k+m+1} \quad \forall n \geq n_0.$$

An upper bound for $(\mu_+ - 1)/(\mu - 1)$ can be obtained in a similar way.

For the second factor, we have

$$\frac{\mu^3}{\mu_+^3} = \left(1 + \frac{2\pi^2}{3\mu^2}\right)^{-3/2}.$$

By a discussion similar to the discussion for $\mu_+/\mu$, we obtain

$$\mu_2 - \varepsilon_2 < \frac{\mu^3}{\mu_+^3} < \mu_2 + \varepsilon_2,$$

where

$$\mu_2 = \sum_{k=0}^{m'} \binom{-3/2}{k} \left(\frac{2\pi^2}{3}\right)^k \mu^{-2k}, \quad \varepsilon_2 = \left|\binom{-3/2}{m'+1}\right| \left(\frac{2\pi^2}{3}\right)^{m'+1} \mu^{-2m'-2},$$

and $m' = \lfloor m/2 \rfloor$.

For the last factor, we firstly substitute $m$ by $m+1$ in (3.1) to get a better estimation. We still use the notation $\mu_1$ and $e_1$ of the estimation of $m+1$. Then

$$e^{\mu(\mu_1-1-\varepsilon_1)} < e^{\mu_+-\mu} < e^{\mu(\mu_1-1+\varepsilon_1)}.$$

Noting that

$$e^{-x} > 1-x, \quad e^x < 1+2x, \quad \forall\, 0 < x < \frac{1}{2},$$

and

$$\sum_{k=0}^{m} \frac{x^k}{k!} < e^x < \sum_{k=0}^{m} \frac{x^k}{k!} + e^x \frac{x^{m+1}}{(m+1)!} \quad \forall\, x > 0,$$

we thus derive that

$$\left(1-\mu\varepsilon_1\right) \cdot \sum_{k=0}^{m} \frac{(\mu(\mu_1-1))^k}{k!} < e^{\mu_+-\mu},$$

and

$$e^{\mu_+-\mu} < \left(1+2\mu\varepsilon_1\right) \cdot \left(\sum_{k=0}^{m} \frac{(\mu(\mu_1-1))^k}{k!} + e^{\mu(\mu_1-1)} \frac{(\mu(\mu_1-1))^{m+1}}{(m+1)!}\right).$$

To get an upper bound for $e^{\mu(\mu_1-1)}$, we use the inequality

$$\mu(\mu_1-1) < \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} \widetilde{\binom{1/2}{k}} \left(\frac{2\pi^2}{3}\right)^k \mu_0^{-2k+1} \quad \forall\, n \geq n_0,$$

where $\tilde{x} = x$ if $x > 0$ and $\tilde{x} = 0$ otherwise.

Combining all the three factors together, we will get an estimation of the ratio $T(n+1)/T(n)$ and thus an estimation of the ratio $p(n+1)/p(n)$.

**Theorem 3.1** *Let $p(n)$ be the partition function. Then for any positive integer $m$, there exist integer $N$, real numbers $a_k$ and $C_1, C_2 > 0$ such that*

$$\sum_{k=0}^{m} a_k \mu^{-k} - C_1 \mu^{-m-1} < \frac{p(n+1)}{p(n)} < \sum_{k=0}^{m} a_k \mu^{-k} + C_2 \mu^{-m-1} \quad \forall\, n \geq N.$$

*Proof* By Theorem 2.2, for any $m$ there exists $N$ such that

$$|p(n)/T(n) - 1| < 2^{m+1}\mu(n)^{-m} \quad \forall n \geq N.$$

That is

$$T(n)(1 - 2^{m+1}\mu(n)^{-m}) < p(n) < T(n)(1 + 2^{m+1}\mu(n)^{-m}).$$

Since $\mu(n)$ is an increasing function of $n$, we thus derive that

$$\frac{T(n+1)}{T(n)}\frac{1 - 2^{m+1}\mu(n)^{-m}}{1 + 2^{m+1}\mu(n)^{-m}} < r_n < \frac{T(n+1)}{T(n)}\frac{1 + 2^{m+1}\mu(n)^{-m}}{1 - 2^{m+1}\mu(n)^{-m}},$$

where $r_n = p(n+1)/p(n)$.

One can check that for $0 < \varepsilon < 1/3$ we have

$$\frac{1+\varepsilon}{1-\varepsilon} < 1 + 3\varepsilon \quad \text{and} \quad \frac{1-\varepsilon}{1+\varepsilon} < 1 - 2\varepsilon.$$

Noting that $0 < \frac{2^{m+1}}{\mu(n)^m} < 1/3$ for any $m \geq 0$, we obtain

$$\frac{T(n+1)}{T(n)}\left(1 - 4 \cdot 2^m \mu(n)^{-m}\right) < r_n < \frac{T(n+1)}{T(n)}\left(1 + 6 \cdot 2^m \mu(n)^{-m}\right), \quad n \geq N. \tag{3.2}$$

Now we consider the three factors of the ratio $\frac{T(n+1)}{T(n)} = \frac{\mu_+ - 1}{\mu - 1} \cdot \frac{\mu^3}{\mu_+^3} \cdot e^{\mu_+ - \mu}$. Each of these factors is bounded by a pair of polynomials in $\mu(n)^{-1}$ as shown in previous paragraphs. Moreover, the difference of the pair of polynomials is a polynomial in $\mu(n)^{-1}$ of degree at least $m+1$. Hence $T(n+1)/T(n)$ and $r_n$ are bounded by a pair of polynomials in $\mu(n)^{-1}$ whose difference is a polynomials in $\mu(n)^{-1}$ with degree at least $m+1$. Since $\lim_{n\to\infty} \mu(n) = +\infty$, the difference is bounded by $C\mu(n)^{-m-1}$ for some constant $C > 0$, completing the proof. □

We have implemented a `Mathematica` package `pn.m` which is accessible at the first named author's homepage to compute these parameters. For example, we compute that

$$f_4(\mu) - \frac{151}{\mu^5} < \frac{p(n+1)}{p(n)} < f_4(\mu) + \frac{419}{\mu^5} \quad \forall n \geq 99,$$

where

$$f_4(\mu) = 1 + \frac{\pi^2}{3\mu} - \frac{\frac{2\pi^2}{3} - \frac{\pi^4}{18}}{\mu^2} - \frac{-\frac{\pi^2}{3} + \frac{5\pi^4}{18} - \frac{\pi^6}{162}}{\mu^3} + \frac{\frac{\pi^2}{3} + \frac{5\pi^4}{9} - \frac{\pi^6}{18} + \frac{\pi^8}{1944}}{\mu^4}.$$

## 4 Asymptotic $r$-log-concavity of $p(n)$

In this section, we will show the asymptotic $r$-log-concavity of $p(n)$ and present a method to find the explicit $N$ such that $\{p(n)\}_{n\geq N}$ is $r$-log-concave.

By Theorem 3.1, we see that to get an estimation of $\mathscr{R}^2 p(n) = p(n)p(n+2)/p(n+1)^2$, we need to consider the bounds of $\mu_+^{-r}$ and $1/f(\mu^{-1})$, where $f(x)$ is a polynomial in $x$ with constant term 1.

Noting that

$$\frac{\mu_+^{-r}}{\mu^{-r}} = \left(1+\frac{2\pi^2}{3\mu^2}\right)^{-r/2},$$

we have an estimation of $\mu_+^{-r}$ up to $\mu^{-m}$ for any positive integer $m$ in a way similar to the estimation of $\mu_+$. To get the bounds of $1/f(\mu^{-1})$, we consider the coefficient $c$ of the tail term of $f(x)-1$. If $c>0$, we will compute an integer $N$ such that

$$f(\mu^{-1})-1>0 \quad \forall n\geq N.$$

Then

$$1-\varepsilon+\varepsilon^2-\cdots+(-1)^m\varepsilon^m-\varepsilon^{m+1} < \frac{1}{f(\mu^{-1})} < 1-\varepsilon+\varepsilon^2-\cdots +(-1)^m\varepsilon^m+\varepsilon^{m+1},$$

where $\varepsilon = f(\mu^{-1})-1$ is a polynomial in $\mu^{-1}$. If $c<0$, we will compute an integer $N$ such that

$$\frac{1}{2} > 1-f(\mu^{-1}) > 0 \quad \forall n\geq N.$$

Then

$$1+\varepsilon+\varepsilon^2+\cdots+\varepsilon^m < \frac{1}{f(\mu^{-1})} < 1+\varepsilon+\varepsilon^2+\cdots+\varepsilon^m+2\varepsilon^{m+1},$$

where $\varepsilon = 1-f(\mu^{-1})$.

Based on the above estimations, we can find the upper and lower bounds of $s_n = \mathscr{R}^2 p(n)$ and further $s_n^{(r)} = \mathscr{R}^2\hat{\mathscr{L}}^{r-1}p(n)$ for $r\geq 2$. All these computations have been implemented in a `Mathematica` package `pn.m` which can be downloaded from the homepage of the first named author.

For example, we have

$$1-\frac{\pi^4}{9\mu^3}-\frac{557}{\mu^4} < s_n = \mathscr{R}^2 p(n) < 1-\frac{\pi^4}{9\mu^3}+\frac{512}{\mu^4} \quad \forall n\geq 46.$$

Noting that for $n \geq 341$, it holds that

$$\frac{\pi^4}{9\mu^3} > \frac{512}{\mu^4},$$

we thus derive that $\{p(n)\}_{n\geq 341}$ is log-concavity. By checking the initial values, we reproved the log-concavity of $\{p(n)\}_{n\geq 26}$.

Notice that for any positive integer $m$,

$$s_n = 1 - \frac{\pi}{2\sqrt{6}n^{3/2}} + \cdots + o\left(\frac{1}{n^m}\right).$$

By Theorem 1.3, we are led to the asymptotic $r$-log-concavity of $p(n)$.

**Theorem 4.1** *For any positive integer $r$, there exists an integer $N$ such that $\{p(n)\}_{n\geq N}$ is $r$-log-concave.*

To find the explicit $N$ such that $\{p(n)\}_{n\geq N}$ is $r$-log-concave, we need to check many initial values. For example, we need the estimation of $s_n$ up to $\mu^{-6}$ to derive the 2-log-concavity.

Denote $s_n^{(i+1)} = \hat{\mathscr{L}}^i a_n$, for $n \geq 281$ we have

$$s_n^{(1)} \geq 1 - \frac{\pi^4}{9\mu^3} + \frac{4\pi^4}{9\mu^4} + \frac{-\frac{\pi^4}{3} + \frac{\pi^6}{9}}{\mu^5} + \frac{-\frac{4\pi^4}{9} - \frac{16\pi^6}{27} + \frac{\pi^8}{162}}{\mu^6} - \frac{2275}{\mu^7}$$

and

$$s_n^{(1)} \leq 1 - \frac{\pi^4}{9\mu^3} + \frac{4\pi^4}{9\mu^4} + \frac{-\frac{\pi^4}{3} + \frac{\pi^6}{9}}{\mu^5} + \frac{-\frac{4\pi^4}{9} - \frac{16\pi^6}{27} + \frac{\pi^8}{162}}{\mu^6} + \frac{11897}{\mu^7},$$

which leads to

$$1 - \frac{2\pi^2}{9\mu^3} - \frac{6303}{\mu^4} \leq s_n^{(2)} \leq 1 - \frac{2\pi^2}{9\mu^3} + \frac{11897}{\mu^4}, \quad \forall n \geq 281.$$

To ensure that $s^{(2)} \leq 1$, we need

$$\frac{2\pi^2}{9\mu^3} \geq \frac{11897}{\mu^4}$$

which holds for $n \geq 24860$. By checking the first 24860 terms, we finally derive that $\{p(n)\}_{n\geq 221}$ is 2-log-concave.

To prove the 3-log-concavity, we need to check about $1.31 \times 10^7$ terms.

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