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Determinantal inequalities for the partition function

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Let $p(n)$ denote the partition function. In this paper, we will prove that for $n \geqslant 222$,

$$\begin{vmatrix} p(n) & p(n+1) & p(n+2) \\ p(n-1) & p(n) & p(n+1) \\ p(n-2) & p(n-1) & p(n) \end{vmatrix} > 0.$$

As a corollary, we deduce that $p(n)$ satisfies the double Turán inequalities, that is, for $n \geqslant 222$,

$$(p(n)^2 - p(n-1)p(n+1))^2 - (p(n-1)^2 - p(n-2)p(n))(p(n+1)^2 - p(n)p(n+2)) > 0.$$

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1. Introduction

The subject of this paper is concerned with the positivity of certain matrices having partition function entries. First, let us recall some background.

Let $\{a_k\}_{k=0}^{n}$ be a sequence of nonnegative real numbers. It is said that $\{a_k\}_{k=0}^{n}$ is *log-concave* (*log-convex*, resp.) if $a_{k+1}^2 \geqslant a_k a_{k+2}$ ($a_{k+1}^2 \leqslant a_k a_{k+2}$, resp.) for all $k \geqslant 0$. For a finite sequence $\{a_k\}_{k=0}^{n}$, it is also called log-concave (log-convex, resp.) if $a_{k+1}^2 \geqslant a_k a_{k+2}$ ($a_{k+1}^2 \leqslant a_k a_{k+2}$, resp.) for all $0 \leqslant k \leqslant n-2$. If there exists an index $0 \leqslant j \leqslant n$ such that $a_k \leqslant a_{k+1}$ for all $k = 0, \ldots, j-1$ and $a_k \geqslant a_{k+1}$ for $k = j, \ldots, n-1$, we call $\{a_n\}_{k=0}^{n}$ a unimodal sequence and if there are not three indices $0 \leqslant k < j < i \leqslant n$ such that a_i, $a_k \neq 0$ and $a_j = 0$, the sequence is said to have no internal zeros.

We say that a real polynomial $P(x) = \sum_{k=0}^{n} a_k x^k$ is log-concave (unimodal, with no internal zeros, resp.) if its the coefficient sequence has the corresponding property. There are many log-concave sequences or unimodal sequences in combinatorics. It has been proved that the coefficients of a real polynomial are log-concave if this polynomial has only real zeros, see [**2**].

THEOREM 1.1. *Let $P(x) = \sum_{k=0}^{n} a_k x^k$ be a real polynomial with nonnegative coefficients and with only real zeros. Then the coefficient sequence $\{a_k\}_{k=0}^{n}$ is log-concave with no internal zeros; in particular, it is unimodal.*

Clearly, the above theorem gives a necessary condition for a real polynomial to have only real zeros.

Given a sequence $\{a_k\}_{k=0}^{\infty}$, we define its Toeplitz matrix T and Hankel matrix H by

$$T = \begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{1.1}$$

and

$$H = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ldots \end{bmatrix}. \tag{1.2}$$

Recall that a real matrix A is totally positive (TP for short) if all its minors are positive. Let r be a positive integer. It is called TP_r if all minors of order $\leqslant r$ are nonnegative. We say that a real infinite sequence $\{a_k\}_{k=0}^{\infty}$ is called a *totally positive sequence* (TP-sequence for short) or *Pólya frequency sequence* (PF-sequence for short) if its Toeplitz matrix is totally positive. A sequence $\{a_k\}_{k=0}^{\infty}$ as above is called a *totally positive sequence of order* r (TP_r-sequence for short) or a Pólya frequency sequence of order r (PF_r-sequence for short) if its Toeplitz matrix is a TP_r matrix. We also call a real finite sequence $a_0, a_1, \ldots, a_n$ a PF-sequence (PF_r-sequence, resp.) if the infinite sequence $a_0, a_1, \ldots, a_n, 0, 0, \ldots$ has the corresponding property. Furthermore, it is easy to see that a sequence of positive numbers is log-concave (log-convex, resp.) if and only if the corresponding Toeplitz matrix (Hankle matrix, resp.) is TP_2.

Aissen, Edrei, Schoenberg and Whitney [**1**] proved the following result.

THEOREM 1.2. *Let $P(x) = \sum_{k=0}^{n} a_k x^k$ be a real polynomial with nonnegative coefficients. Then $P(x)$ has only real zeros if and only if its coefficient sequence is a PF-sequence.*

A sequence $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ of real numbers is called a *multiplier sequence* if, wherever the real polynomial $P(x) = \sum_{k=0}^{n} a_k x^k$ has only real zeros, the polynomial $\Gamma(P(x)) = \sum_{k=0}^{n} \gamma_k a_k x^k$ also has only real zeros. This theory commenced with the work of Laguerre [**12**] and was solidified in the seminal work of Pólya and Schur [**14**]. Multiplier sequences have been extensively studied in the theory of total positivity [**11**] and in combinatorics [**2**].

Recall that a real entire function

$$\psi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \tag{1.3}$$

is said to be in the *Laguerre-Pólya class*, denoted $\psi(x) \in \mathcal{LP}$, if it can be represented in the form

$$\psi(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} (1 + x/x_k)\, e^{-x/x_k},$$

where c, β, x_k are real numbers, $\alpha \geqslant 0$, m is a nonnegative integer and $\sum x_k^{-2} < \infty$. These functions are the only ones which are uniform limits of polynomials whose zeros are real. If $\gamma_k \geqslant 0$ (or $(-1)^k \gamma_k \geqslant 0$ or $\gamma_k \leqslant 0$) for all $k = 0, 1, 2, \ldots,$ then $\psi(x) \in \mathcal{LP}$ is said to be of *type I* in the Laguerre-Pólya class written as $\psi(x) \in \mathcal{LPI}$. It is well-known that a sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence if and only if its exponential generating function $P(x) = \sum_{k=0}^{\infty} \gamma_k/k! x^k$ belongs to $\mathcal{LPI}$. We refer to [**13**, **15**] for the background on the theory of the $\mathcal{LP}$ class.

It is easy to find an intimate connection between multiplier sequences and PF-sequences from the following theorem.

Theorem 1.3 (Aissen *et al.* [**1**]). *Let* $\psi(x) = \sum_{k=0}^{\infty} \alpha_k x^k$, $\alpha_0 = 1$, $\alpha_k \in \mathbb{R}^+$, *be an entire function. Then* $\{\alpha_k\}_{k=0}^{\infty}$ *is a* PF*-sequence if and only if* $\psi(x) \in \mathcal{LPI}$.

Concerning multiplier sequences, Craven and Csordas [**6**] obtained the following theorem.

Theorem 1.4 (Theorem 2.13, [**6**]). *If* $\{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k > 0$, *is a multiplier sequence, then*

$$\begin{vmatrix} \gamma_k & \gamma_{k+1} & \gamma_{k+2} \\ \gamma_{k-1} & \gamma_k & \gamma_{k+1} \\ \gamma_{k-2} & \gamma_{k-1} & \gamma_k \end{vmatrix} \geqslant 0, \quad \textit{for } k = 2, 3, 4, \ldots . \tag{1.4}$$

Notice that the left-hand side of (1.4) is equivalent to

$$\frac{1}{\gamma_k} \left((\gamma_k^2 - \gamma_{k-1}\gamma_{k+1})^2 - (\gamma_{k-1}^2 - \gamma_{k-2}\gamma_k)(\gamma_{k+1}^2 - \gamma_k \gamma_{k+2}) \right).$$

Thus theorem 1.4 gives us

$$(\gamma_k^2 - \gamma_{k-1}\gamma_{k+1})^2 - (\gamma_{k-1}^2 - \gamma_{k-2}\gamma_k)(\gamma_{k+1}^2 - \gamma_k \gamma_{k+2}) \geqslant 0,$$

which is called a *double Turán inequality*, see [**6**–**8**].

Let us now turn to the partition function. A partition of a positive integer n is a nonincreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ of positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. Let $p(n)$ denote the number of partitions of n. DeSalvo and Pak [**9**] proved the log-concavity of the partition function for $n \geqslant 26$ as well as the following inequality as conjectured in [3]:

$$\frac{p(n-1)}{p(n)}\left(1+\frac{1}{n}\right) > \frac{p(n)}{p(n+1)}, \quad n \geqslant 2. \tag{1.5}$$

DaSalvo and Pak also conjectured that for $n \geqslant 45$,

$$\frac{p(n-1)}{p(n)}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{p(n)}{p(n+1)}. \tag{1.6}$$

Chen, Wang and Xie [5] gave an affirmative answer to this conjecture.

Recently, Chen and Jia and Wang [4] showed that for $n \geqslant 95$, $p(n)$ satisfies a high order Turán inequality, that is,

$$\begin{aligned} &4(p(n)^2 - p(n-1)p(n+1))(p(n+1)^2 - p(n)p(n+2)) \\ &\quad - (p(n)p(n+1) - p(n-1)p(n+2))^2 \geqslant 0 \end{aligned} \tag{1.7}$$

holds for $n \geqslant 95$.

In the remaining of this paper, we shall prove the following result.

THEOREM 1.5. *Let $p(n)$ denote the partition function and*

$$M_3(p(n)) = \begin{pmatrix} p(n) & p(n+1) & p(n+2) \\ p(n-1) & p(n) & p(n+1) \\ p(n-2) & p(n-1) & p(n) \end{pmatrix} \tag{1.8}$$

Then for $n \geqslant 222$, we have

$$\det M_3(p(n)) > 0. \tag{1.9}$$

Note that 222 is best possible since $\det M_3(p(221)) < 0$. In fact, $\det M_3(p(n))$ is negative for all odd n with $3 \leqslant n \leqslant 221$.

From the above theorem, we immediately get the following result.

THEOREM 1.6. *The partition function $p(n)$ satisfies the double Turán inequalities for $n \geqslant 222$, that is, for $n \geqslant 222$,*

$$\begin{aligned} &\big(p(n)^2 - p(n-1)p(n+1)\big)^2 - \big(p(n-1)^2 - p(n-2)p(n)\big) \\ &\quad \times \big(p(n+1)^2 - p(n)p(n+2)\big) > 0. \end{aligned}$$

Remark that recently, Hou and Zhang [10] independently proved theorem 1.6 by a different approach.

Let

$$M_k(p(n)) = (p(n-i+j))_{1 \leqslant i,j \leqslant k}.$$

Numerical evidences show that for all $656 \leqslant n \leqslant 10\,000$, $\det(M_4(p(n))) > 0$ and for all $1372 \leqslant n \leqslant 10\,000$ $\det(M_5(p(n))) > 0$. These facts encourage us to propose the following conjecture.

CONJECTURE 1.7. *For any given k, there exists a positive integer $n(k)$ such that for $n > n(k)$,*

$$\det(M_k(p(n))) > 0.$$

2. Upper bound for $s(n)$

Let

$$u_n = \frac{p(n-1)p(n+1)}{p(n)^2}. \tag{2.1}$$

It can be checked that

$$\det M_3(p(n)) = p(n)^3 \left(u_n^2 \left(u_{n-1} + u_{n+1} - u_{n-1}u_{n+1}\right) - 2u_n + 1\right). \tag{2.2}$$

For convenience, we denote

$$s(n) = u_{n-1} + u_{n+1} - u_{n-1}u_{n+1}, \tag{2.3}$$

and hence, theorem 1.5 can be restated as follows.

THEOREM 2.1. *Let u_n be defined as in (2.1) and $s(n)$ be denoted as in (2.3). Then for $n > 221$, we have*

$$s(n)u_n^2 - 2u_n + 1 > 0. \tag{2.4}$$

In this section, we shall give an upper and a lower bound for $s(n)$, which plays an important role in (2.4).

To this aim, we denote

$$\mu(n) = \frac{\pi\sqrt{24n-1}}{6}.$$

Let $r = \mu(n-2)$ and adopt the following notation as used in [4]:

$$x = \mu(n-1), \quad y = \mu(n), \quad z = \mu(n+1), \quad w = \mu(n+2), \tag{2.5}$$

and

$$f(n) = e^{x-2y+z} \frac{\left(x^{10} - x^9 - 1\right) y^{24} \left(z^{10} - z^9 - 1\right)}{x^{12} \left(y^{10} - y^9 + 1\right)^2 z^{12}}, \tag{2.6}$$

$$g(n) = e^{x-2y+z} \frac{\left(x^{10} - x^9 + 1\right) y^{24} \left(z^{10} - z^9 + 1\right)}{x^{12} \left(y^{10} - y^9 - 1\right)^2 z^{12}}. \tag{2.7}$$

Employing Rademacher's convergent series and Lehmer's error bound, Chen, Jia and Wang [4] proved the following inequality.

THEOREM 2.2. *For $n \geqslant 1207$,*

$$f(n) < u_n < g(n). \tag{2.8}$$

Set

$$s_1(n) = f(n-1) + f(n+1) - g(n-1)g(n+1), \tag{2.9}$$

and

$$s_2(n) = g(n-1) + g(n+1) - f(n-1)f(n+1). \tag{2.10}$$

By the above theorem, we get the following bounds for $s(n)$.

Corollary 2.3. *For $n \geqslant 1207$, we have*

$$s_1(n) < s(n) < s_2(n). \tag{2.11}$$

The upper bound $s_2(n)$ for $s(n)$ is very precise and to make our proof easy, we claim that $s(n) < 1$ for $n \geqslant 1207$. In fact, we have the following theorem.

Theorem 2.4. *Let $s_2(n)$ be defined as in (2.10). Then for $n \geqslant 3$, we have*

$$s_2(n) < 1. \tag{2.12}$$

Proof. Recall that

$$s_2(n) = g(n-1) + g(n+1) - f(n-1)f(n+1), \tag{2.13}$$

and hence, to verify (2.12), we estimate to $g(n-1)$, $g(n+1)$, $f(n-1)$ and $f(n+1)$ first. For this aim, we prefer to give the estimates of r, x, z and w by the following equalities: For $n \geqslant 3$,

$$r = \sqrt{y^2 - \frac{4\pi^2}{3}}, \quad x = \sqrt{y^2 - \frac{2\pi^2}{3}}, \quad z = \sqrt{y^2 + \frac{2\pi^2}{3}}, \quad w = \sqrt{y^2 + \frac{4\pi^2}{3}}. \tag{2.14}$$

Following Chen, Jia and Wang [4], we obtain the following expansions easily:

$$r = y - \frac{2\pi^2}{3y} - \frac{2\pi^4}{9y^3} - \frac{4\pi^6}{27y^5} - \frac{10\pi^8}{81y^7} - \frac{28\pi^{10}}{243y^9} - \frac{28\pi^{12}}{243y^{11}} - \frac{88\pi^{14}}{729y^{13}} + O\left(\frac{1}{y^{15}}\right),$$

$$x = y - \frac{\pi^2}{3y} - \frac{\pi^4}{18y^3} - \frac{\pi^6}{54y^5} - \frac{5\pi^8}{648y^7} - \frac{7\pi^{10}}{1944y^9} - \frac{7\pi^{12}}{3888y^{11}} - \frac{11\pi^{14}}{11664y^{13}} + O\left(\frac{1}{y^{15}}\right),$$

$$z = y + \frac{\pi^2}{3y} - \frac{\pi^4}{18y^3} + \frac{\pi^6}{54y^5} - \frac{5\pi^8}{648y^7} + \frac{7\pi^{10}}{1944y^9} - \frac{7\pi^{12}}{3888y^{11}} + \frac{11\pi^{14}}{11664y^{13}} + O\left(\frac{1}{y^{15}}\right),$$

and

$$w = y + \frac{2\pi^2}{3y} - \frac{2\pi^4}{9y^3} + \frac{4\pi^6}{27y^5} - \frac{10\pi^8}{81y^7} + \frac{28\pi^{10}}{243y^9} - \frac{28\pi^{12}}{243y^{11}} + \frac{88\pi^{14}}{729y^{13}} + O\left(\frac{1}{y^{15}}\right).$$

It is readily checked that for $n \geqslant 145$,

$$r_1 < r < r_2, \tag{2.15}$$

$$x_1 < x < x_2, \tag{2.16}$$

$$z_1 < z < z_2, \tag{2.17}$$

$$w_1 < w < w_2, \tag{2.18}$$

where

$$r_1 = y - \frac{2\pi^2}{3y} - \frac{2\pi^4}{9y^3} - \frac{4\pi^6}{27y^5} - \frac{10\pi^8}{81y^7} - \frac{28\pi^{10}}{243y^9} - \frac{28\pi^{12}}{243y^{11}} - \frac{89\pi^{14}}{729y^{13}},$$

$$r_2 = y - \frac{2\pi^2}{3y} - \frac{2\pi^4}{9y^3} - \frac{4\pi^6}{27y^5} - \frac{10\pi^8}{81y^7} - \frac{28\pi^{10}}{243y^9} - \frac{28\pi^{12}}{243y^{11}} - \frac{88\pi^{14}}{729y^{13}},$$

$$x_1 = y - \frac{\pi^2}{3y} - \frac{\pi^4}{18y^3} - \frac{\pi^6}{54y^5} - \frac{5\pi^8}{648y^7} - \frac{7\pi^{10}}{1944y^9} - \frac{7\pi^{12}}{3888y^{11}} - \frac{12\pi^{14}}{11664y^{13}},$$

$$x_2 = y - \frac{\pi^2}{3y} - \frac{\pi^4}{18y^3} - \frac{\pi^6}{54y^5} - \frac{5\pi^8}{648y^7} - \frac{7\pi^{10}}{1944y^9} - \frac{7\pi^{12}}{3888y^{11}} - \frac{11\pi^{14}}{11664y^{13}},$$

$$z_1 = y + \frac{\pi^2}{3y} - \frac{\pi^4}{18y^3} + \frac{\pi^6}{54y^5} - \frac{5\pi^8}{648y^7} + \frac{7\pi^{10}}{1944y^9} - \frac{7\pi^{12}}{3888y^{11}},$$

$$z_2 = y + \frac{\pi^2}{3y} - \frac{\pi^4}{18y^3} + \frac{\pi^6}{54y^5} - \frac{5\pi^8}{648y^7} + \frac{7\pi^{10}}{1944y^9} - \frac{7\pi^{12}}{3888y^{11}} + \frac{11\pi^{14}}{11664y^{13}},$$

$$w_1 = y + \frac{2\pi^2}{3y} - \frac{2\pi^4}{9y^3} + \frac{4\pi^6}{27y^5} - \frac{10\pi^8}{81y^7} + \frac{28\pi^{10}}{243y^9} - \frac{28\pi^{12}}{243y^{11}},$$

$$w_2 = y + \frac{2\pi^2}{3y} - \frac{2\pi^4}{9y^3} + \frac{4\pi^6}{27y^5} - \frac{10\pi^8}{81y^7} + \frac{28\pi^{10}}{243y^9} - \frac{28\pi^{12}}{243y^{11}} + \frac{88\pi^{14}}{729y^{13}}.$$

Applying these bounds for r, x, z and w, we now estimate $g(n-1)$, $g(n+1)$, $f(n-1)$ and $f(n+1)$. For convenience, set

$$h(n) = \frac{\left(x^{10} - x^9 - 1\right) y^{24} \left(z^{10} - z^9 - 1\right)}{x^{12} \left(y^{10} - y^9 + 1\right)^2 z^{12}} \tag{2.19}$$

and

$$q(n) = \frac{\left(x^{10} - x^9 + 1\right) y^{24} \left(z^{10} - z^9 + 1\right)}{x^{12} \left(y^{10} - y^9 - 1\right)^2 z^{12}}. \tag{2.20}$$

Then,

$$f(n) = e^{x-2y+z} h(n), \quad g(n) = e^{x-2y+z} q(n), \tag{2.21}$$

which suggest that we should bound $f(n-1)$, $f(n+1)$, $g(n-1)$ and $g(n+1)$ by estimating e^{r-2x+y}, e^{y-2z+w}, $h(n-1)$, $h(n+1)$, $q(n-1)$ and $q(n+1)$ separately.

We first consider the exponential factors e^{r-2x+y} and e^{y-2z+w}. It is easily seen that for $n \geqslant 145$

$$r_1 - 2x_2 + y < r - 2x + y < r_2 - 2x_1 + y, \tag{2.22}$$

$$y - 2z_2 + w_1 < y - 2z + w < y - 2z_1 + w_2, \tag{2.23}$$

which implies that

$$e^{r_1 - 2x_2 + y} < e^{r-2x+y} < e^{r_2 - 2x_1 + y}, \tag{2.24}$$

$$e^{y - 2z_2 + w_1} < e^{y-2z+w} < e^{y - 2z_1 + w_2}. \tag{2.25}$$

In order to give a feasible bound for e^{r-2x+y} and e^{y-2z+w}, we define

$$\Phi(t)=1+t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}+\frac{t^6}{720}, \tag{2.26}$$

and

$$\phi(t)=1+t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}+\frac{t^6}{720}+\frac{t^7}{5040}. \tag{2.27}$$

It can be checked that for $t<0$,

$$\phi(t)<e^t<\Phi(t). \tag{2.28}$$

To apply this result to (2.24) and (2.25), we have to show that both r_2-2x_1+y and $y-2z_1+w_2$ are negative. By straightforward calculation, it is easy to obtain that

$$r_2-2x_1+y=-\frac{\pi^4\left(648y^{10}+648\pi^2y^8+630\pi^4y^6+630\pi^6y^4+651\pi^8y^2+692\pi^{10}\right)}{5832y^{13}},$$

which is clearly negative. As for $y-2z_1+w_2$, we find that

$$y-2z_1+w_2=-\frac{\pi^4\left(648y^{10}-648\pi^2y^8+630\pi^4y^6-630\pi^6y^4+651\pi^8y^2+11\pi^{10}\right)}{5832y^{13}},$$

which is also negative for $n\geqslant 2$. This fact can be confirmed by noting that for $n\geqslant 2$,

$$\begin{aligned}&648\pi^2y^8-648y^{10}<0,\\&630\pi^6y^4-630\pi^4y^6<0,\\&704\pi^{10}-651\pi^8y^2<0.\end{aligned}$$

Thus, applying (2.28) to (2.24) and (2.25), we obtain that for $n\geqslant 145$,

$$\phi(r_1-2x_2+y)<e^{r-2x+y}<\Phi(r_2-2x_1+y), \tag{2.29}$$

and

$$\phi(y-2z_2+w_1)<e^{y-2z+w}<\Phi(y-2z_1+w_2). \tag{2.30}$$

Now we turn to estimate $h(n-1)$, $h(n+1)$, $q(n-1)$ and $q(n+1)$.

Let

$$\alpha(t) = t^{10} - t^9 + 1, \quad \beta(t) = t^{10} - t^9 - 1. \tag{2.31}$$

With these notation, $h(n-1)$, $h(n+1)$, $q(n-1)$ and $q(n+1)$ can be rewritten as

$$h(n-1) = \frac{x^{24}\beta(r)\beta(y)}{r^{12}y^{12}\alpha(x)^2}, \quad h(n+1) = \frac{z^{24}\beta(y)\beta(w)}{w^{12}y^{12}\alpha(z)^2},$$

$$q(n-1) = \frac{x^{24}\alpha(r)\alpha(y)}{r^{12}y^{12}\beta(x)^2}, \quad q(n+1) = \frac{z^{24}\alpha(y)\alpha(w)}{y^{12}w^{12}\beta(z)^2}.$$

For our purpose, we recall (2.15), (2.16), (2.17), (2.18) and establish the following inequalities, which can be directly checked, that is, for $n \geqslant 145$,

$$\begin{gathered}
r^{10} - r_2 r^8 + 1 < \alpha(r) < r^{10} - r_1 r^8 + 1,\\
x^{10} - x_2 x^8 + 1 < \alpha(x) < x^{10} - x_1 x^8 + 1,\\
z^{10} - z_2 z^8 + 1 < \alpha(z) < z^{10} - z_1 z^8 + 1,\\
w^{10} - w_2 w^8 + 1 < \alpha(w) < w^{10} - w_1 w^8 + 1,\\
r^{10} - r_1 r^8 - 1 < \beta(r) < r^{10} - r_2 r^8 - 1,\\
x^{10} - x_1 x^8 - 1 < \beta(x) < x^{10} - x_2 x^8 - 1,\\
z^{10} - z_1 z^8 - 1 < \beta(z) < z^{10} - z_2 y^8 - 1,\\
w^{10} - w_1 w^8 - 1 < \beta(w) < w^{10} - w_2 w^8 - 1,\\
x^{20} - 2x_2 x^{18} + x^{18} + 2x^{10} - 2x_2 x^8 + 1 < \alpha(x)^2 < x^{20} - 2x_1 x^{18} + x^{18} + 2x^{10} - 2x_1 x^8 + 1,\\
z^{20} - 2z_2 z^{18} + z^{18} + 2z^{10} - 2z_2 z^8 + 1 < \alpha(z)^2 < z^{20} - 2z_1 z^{18} + z^{18} + 2z^{10} - 2z_1 z^8 + 1,\\
x^{20} - 2x_2 x^{18} + x^{18} - 2x^{10} + 2x_1 x^8 + 1 < \beta(x)^2 < x^{20} - 2x_1 x^{18} + x^{18} - 2x^{10} + 2x_2 x^8 + 1,\\
z^{20} - 2z_2 z^{18} + z^{18} - 2z^{10} + 2z_1 z^8 + 1 < \beta(z)^2 < z^{20} - 2z_1 z^{18} + z^{18} - 2z^{10} + 2z_2 z^8 + 1.
\end{gathered} \tag{2.32}$$

Applying these above inequalities to $h(n-1)$, $h(n+1)$, $q(n-1)$ and $q(n+1)$ leads to

$$h(n-1) > \frac{\left(r^{10} - r_2 r^8 - 1\right) x^{24} \left(y^{10} - y^9 - 1\right)}{r^{12}\left(x^{20} - 2x_1 x^{18} + x^{18} + 2x^{10} - 2x_1 x^8 + 1\right) y^{12}}, \tag{2.33}$$

$$h(n+1) > \frac{\left(w^{10} - w_2 w^8 - 1\right)\left(y^{10} - y^9 - 1\right) z^{24}}{w^{12} y^{12}\left(z^{20} - 2z_1 z^{18} + z^{18} + 2z^{10} - 2z_1 z^8 + 1\right)}, \tag{2.34}$$

$$q(n-1) < \frac{\left(r^{10} - r_1 r^8 + 1\right) x^{24} \left(y^{10} - y^9 + 1\right)}{r^{12}\left(x^{20} - 2x_2 x^{18} + x^{18} - 2x^{10} + 2x_1 x^8 + 1\right) y^{12}}, \tag{2.35}$$

$$q(n+1) < \frac{\left(w^{10} - w_1 w^8 + 1\right)\left(y^{10} - y^9 + 1\right) z^{24}}{w^{12} y^{12}\left(z^{20} - 2z_2 z^{18} + z^{18} - 2z^{10} + 2z_1 z^8 + 1\right)}. \tag{2.36}$$

Now, combining (2.21), (2.29) and (2.30), we can bound $f(n-1)$, $f(n+1)$, $g(n-1)$ and $g(n+1)$ easily, that is, for $n \geqslant 145$,

$$g(n-1) < \lambda_1(y) = \frac{\left(r^{10} - r_1 r^8 + 1\right) x^{24} \left(y^{10} - y^9 + 1\right) \Phi(r_2 - 2x_1 + y)}{r^{12} \left(x^{20} - 2x_2 x^{18} + x^{18} - 2x^{10} + 2x_1 x^8 + 1\right) y^{12}},$$

$$g(n+1) < \lambda_2(y) = \frac{\left(w^{10} - w_1 w^8 + 1\right) \left(y^{10} - y^9 + 1\right) z^{24} \Phi(y - 2z_1 + w_2)}{w^{12} y^{12} \left(z^{20} - 2z_2 z^{18} + z^{18} - 2z^{10} + 2z_1 z^8 + 1\right)},$$

$$f(n-1) > \lambda_3(y) = \frac{\left(r^{10} - r_2 r^8 - 1\right) x^{24} \left(y^{10} - y^9 - 1\right) \phi(r_1 - 2x_2 + y)}{r^{12} \left(x^{20} - 2x_1 x^{18} + x^{18} + 2x^{10} - 2x_1 x^8 + 1\right) y^{12}},$$

$$f(n+1) > \lambda_4(y) = \frac{\left(w^{10} - w_2 w^8 - 1\right) \left(y^{10} - y^9 - 1\right) z^{24} \phi(y - 2z_2 + w_1)}{w^{12} y^{12} \left(z^{20} - 2z_1 z^{18} + z^{18} + 2z^{10} - 2z_1 z^8 + 1\right)}.$$

Since

$$s_2(n) = g(n-1) + g(n+1) - f(n-1) f(n+1),$$

it is easy to get that for $n \geqslant 145$

$$s_2(n) < \lambda_1(y) + \lambda_2(y) - \lambda_3(y)\lambda_4(y). \tag{2.37}$$

We proceed to prove (2.12). In view of (2.37), we only need to show that for $n \geqslant 31072$,

$$\lambda_1(y) + \lambda_2(y) - \lambda_3(y)\lambda_4(y) - 1 < 0. \tag{2.38}$$

The left-hand side in the above inequalities can be simplified as

$$\lambda_1(y) + \lambda_2(y) - \lambda_3(y)\lambda_4(y) - 1 = \frac{H(y)}{G(y)}, \tag{2.39}$$

where $H(y)$ and $G(y)$ are polynomials in y whose degree are separately 356 and 362.

For the convenience of calculation, we show that for $n \geqslant 31072$,

$$H(y)G(y) < 0. \tag{2.40}$$

The left-hand side of (2.40) is a polynomial in y of degree 718, and we write

$$H(y)G(y) = \sum_{k=0}^{718} a_k y^k. \tag{2.41}$$

Here we just list the value of a_{716}, a_{717} and a_{718}:

$$a_{716} = -2^{113} 3^{323} 5^5 7^4 \pi^8,$$

$$a_{717} = 2^{115} 3^{321} 5^4 7^4 \pi^8,$$

$$a_{718} = -2^{112} 3^{320} 5^4 7^4 \pi^8.$$

Noting that y is positive for $n \geqslant 1$, we can easily deduce that

$$H(y)G(y) < \sum_{k=0}^{717} |a_k| y^k + a_{718} y^{718}. \tag{2.42}$$

Moreover, it is easily seen that for any $0 \leqslant k \leqslant 715$,

$$|a_k| y^k < -a_{716} y^{716} \tag{2.43}$$

holds for all $y \geqslant 8$ or $n \geqslant 10$. Thus it follows that for $y \geqslant 8$,

$$\sum_{k=0}^{717} |a_k| y^k + a_{718} y^{718} < (-717 a_{716} + a_{717} y + a_{718} y^2) y^{716}. \tag{2.44}$$

Combining (2.42) and (2.44) leads to

$$H(y)G(y) < (-717 a_{716} + a_{717} y + a_{718} y^2) y^{716} \tag{2.45}$$

for $y \geqslant 8$. Hence, (2.40) holds as long as

$$-717 a_{716} + a_{717} y + a_{718} y^2 < 0, \tag{2.46}$$

which is true if

$$y > 3\left(4 + \sqrt{21526}\right) \approx 452.152.$$

Hence, $H(y)G(y)$ is negative for $y \geqslant 453$ and this implies (2.38).

Combining (2.37) and (2.38), we deduce that for $y \geqslant 453$, or equivalently, for $n \geqslant 31072$, (2.12) is true. Moreover, it can be directly checked for $3 \leqslant n \leqslant 31071$, (2.12) holds. This completes the proof. □

3. An inequality involving $g(n)$ and $s(n)$

In this section, we establish an inequality between $g(n)$ and $s(n)$, which is the key to prove theorem 2.1.

THEOREM 3.1. *Let*

$$\varphi(t) = \frac{1 - \sqrt{1-t}}{t}, \tag{3.1}$$

where $0 < t < 1$. *Then for* $n \geqslant 200$, *we have*

$$g(n) < \varphi(s(n)). \tag{3.2}$$

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Proof. To prove this theorem, we first consider the monotonicity of $\varphi(t)$. Noting that

$$\varphi'(t) = \frac{1}{2\left(\sqrt{1-t}+1\right)^2\sqrt{1-t}}, \tag{3.3}$$

which is positive for $0 < t < 1$, we find that $\varphi(t)$ is increasing for $0 < t < 1$. In view of the monotonicity of $\varphi(t)$ and corollary 2.3, we deduce that for $n \geqslant 1207$,

$$\varphi(s(n)) > \varphi(s_1(n)). \tag{3.4}$$

Thus, to verify (3.2), we turn to show that for $n \geqslant 1207$,

$$g(n) < \varphi(s_1(n)), \tag{3.5}$$

which can be rewritten as

$$s_1(n)g(n)^2 - 2g(n) + 1 > 0. \tag{3.6}$$

By the definition (2.31) of $\alpha(t)$ and $\beta(t)$, the left-hand side of the above inequality can be simplified to

$$\begin{aligned} & s_1(n)g(n)^2 - 2g(n) + 1 \\ &= \frac{-h_1e^{r+w-2y} + h_2e^{w+2x-3y} + h_3 - 2h_4e^{x-2y+z} + h_5e^{r-3y+2z}}{r^{12}w^{12}x^{24}\left(x^{10}-x^9-1\right)^2\left(y^{10}-y^9-1\right)^4 z^{24}\left(z^{10}-z^9-1\right)^2}, \end{aligned} \tag{3.7}$$

where

$$h_1 = x^{24}y^{24}z^{24}\alpha(r)\alpha(w)\alpha(x)^2\alpha(y)^2\alpha(z)^2, \tag{3.8}$$

$$h_2 = r^{12}y^{36}z^{24}\beta(w)\beta(x)^2\alpha(x)^2\beta(y)\beta(z)^2, \tag{3.9}$$

$$h_3 = r^{12}w^{12}x^{24}z^{24}\beta(x)^2\beta(y)^4\beta(z)^2, \tag{3.10}$$

$$h_4 = r^{12}w^{12}x^{12}y^{24}z^{12}\alpha(x)\beta(x)^2\beta(y)^2\alpha(z)\beta(z)^2, \tag{3.11}$$

$$h_5 = w^{12}x^{24}y^{36}\beta(r)\beta(x)^2\beta(y)\beta(z)^2\alpha(z)^2. \tag{3.12}$$

For the convenience, let

$$A(y) = -h_1e^{r+w-2y} + h_2e^{w+2x-3y} + h_3 - 2h_4e^{x-2y+z} + h_5e^{r-3y+2z}. \tag{3.13}$$

Since the denominator of (3.7) is positive, to prove (3.6), we aim to prove the numerator $A(y)$ is positive too. For this aim, we shall estimate e^{r+w-2y}, $e^{w+2x-3y}$, e^{x-2y+z} and $e^{r-3y+2z}$ by the same method used in previous section. First, we apply (2.15), (2.16), (2.17) and (2.18) to the indexes of these functions and obtain that for $n \geqslant 145$,

$$r_1 + w_1 - 2y < r + w - 2y < r_2 + w_2 - 2y, \tag{3.14}$$

$$w_1 + 2x_1 - 3y < w + 2x - 3y < w_2 + 2x_2 - 3y, \tag{3.15}$$

$$x_1 - 2y + z_1 < x - 2y + z < x_2 - 2y + z_2, \tag{3.16}$$

$$r_1 - 3y + 2z_1 < r - 3y + 2z < r_2 - 3y + 2z_2. \tag{3.17}$$

From the monotonicity of exponential function, we find that

$$e^{r+w-2y} < e^{r_2+w_2-2y}, \tag{3.18}$$

$$e^{w+2x-3y} > e^{w_1+2x_1-3y}, \tag{3.19}$$

$$e^{x+z-2y} < e^{x_2+z_2-2y}, \tag{3.20}$$

$$e^{r+2z-3y} > e^{r_1+2z_1-3y}. \tag{3.21}$$

To apply (2.28), we aim to show that for $n \geqslant 145$ the indexes in the right-hand side of the above four inequalities, that is,

$$r_2 + w_2 - 2y = -\frac{4\pi^4\left(27y^8 + 15\pi^4y^4 + 14\pi^8\right)}{243y^{11}},$$

$$w_1 + 2x_1 - 3y = -\frac{\pi^4\left(648y^{10} - 216\pi^2y^8 + 270\pi^4y^6 - 210\pi^6y^4 + 231\pi^8y^2 + 4\pi^{10}\right)}{1944y^{13}},$$

$$x_2 - 2y + z_2 = -\frac{\pi^4\left(216y^8 + 30\pi^4y^4 + 7\pi^8\right)}{1944y^{11}},$$

$$r_1 + 2z_1 - 3y = -\frac{\pi^4\left(1944y^{10} + 648\pi^2y^8 + 810\pi^4y^6 + 630\pi^6y^4 + 693\pi^8y^2 + 712\pi^{10}\right)}{5832y^{13}},$$

are negative. In fact, we only need to verify the second is negative since the others are obviously negative. Noting that for $n \geqslant 145$,

$$648y^{10} - 216\pi^2y^8 > 0,$$
$$270\pi^4y^6 - 210\pi^6y^4 > 0,$$
$$231\pi^8y^2 + 4\pi^{10} > 0,$$

we can easily conclude that $w_1 + 2x_1 - 3y$ is negative, and hence, we now can apply (2.28) to (3.18), (3.19), (3.20) and (3.21) and obtain that for $n \geqslant 145$,

$$e^{r+w-2y} < e^{r_2+w_2-2y} < \Phi(r_2 + w_2 - 2y), \tag{3.22}$$

$$e^{w+2x-3y} > e^{w_1+2x_1-3y} > \phi(w_1 + 2x_1 - 3y), \tag{3.23}$$

$$e^{x+z-2y} < e^{x_2+z_2-2y} < \Phi(x_2 + z_2 - 2y), \tag{3.24}$$

$$e^{r+2z-3y} > e^{r_1+2z_1-3y} > \phi(r_1 + 2z_1 - 3y). \tag{3.25}$$

After applying the estimates of the exponential functions, we have that for $n \geqslant 145$,

$$A > -h_1\Phi(r_2 + w_2 - 2y) + h_2\phi(w_1 + 2x_1 - 3y) + h_3 - 2h_4\Phi(x_2 + z_2 - 2y) + h_5\phi(r_1 + 2z_1 - 3y).$$

Recalling the definition of these polynomials h_1, h_2, h_3, h_4, h_5 and (2.32) and substituting r, x, z, w by the expressions in (2.14), it is easy to obtain a lower bound for the right-hand side of the above inequality, that is,

$$\frac{\sum_{k=0}^{256} a_k y^k}{2^{35}3^{117}35y^{107}}, \tag{3.26}$$

where a_k are known real number, and the value of a_{256}, a_{255}, a_{254} are given below:

$$a_{256} = 2^{37}3^{116}35\pi^2, \quad a_{255} = -2^{38}3^{117}35\pi^2, \quad a_{254} = -2^{37}3^{115}35\pi^2\left(2\pi^2 - 45\right).$$

Thus, for $n \geqslant 145$, we have

$$A > \frac{\sum_{k=0}^{256} a_k y^k}{2^{35}3^{117}35y^{107}}. \tag{3.27}$$

Since $2^{35}3^{117}35y^{107}$ is positive for $n \geqslant 1$, to verify (3.6), we proceed to prove for $n \geqslant 1207$,

$$\sum_{k=0}^{256} a_k y^k > 0. \tag{3.28}$$

As y is positive for $n \geqslant 1$, we see that

$$\sum_{k=0}^{256} a_k y^k > \sum_{k=0}^{255} -|a_k| y^k + a_{256}y^{256}. \tag{3.29}$$

Hence, to verify (3.28), we aim to show that for $n \geqslant 1207$,

$$\sum_{k=0}^{255} -|a_k| y^k + a_{256}y^{256} > 0. \tag{3.30}$$

For $0 \leqslant k \leqslant 253$, we find that

$$-|a_k| y^k > a_{254}y^{254} \tag{3.31}$$

holds for $y > \sqrt{(-270 + 120\pi^2 + 299\pi^4)/(270 - 12\pi^2)} \approx 15$. Therefore, we obtain that for $y \geqslant 16$,

$$\sum_{k=0}^{256} a_k y^k > \sum_{k=0}^{255} -|a_k| y^k + a_{256}y^{256} > (-255a_{254} + a_{255}y + a_{256}y^2)y^{254}. \tag{3.32}$$

It is clear that (3.28) is true provided

$$-255a_{254} + a_{255}y + a_{256}y^2 > 0, \tag{3.33}$$

which holds for $y > \sqrt{3834 - 170\pi^2} + 3 \approx 50$, or equivalently, for $n \geqslant 380$. So we get that (3.6) holds for $n \geqslant 1207$, which implies (3.5). Combining (3.4) and (3.5), we conclude that for $n \geqslant 1207$

$$g(n) < \varphi(s(n)). \tag{3.34}$$

And for $200 \leqslant n \leqslant 1206$, it can be checked that (3.2) is also true, which completes this proof. □

4. Proof of theorem 2.1

In this section, we shall give a proof of theorem 2.1, which states that for $n \geqslant 222$,

$$s(n)u_n^2 - 2u_n + 1 > 0, \tag{4.1}$$

where

$$u_n = \frac{p(n-1)p(n+1)}{p(n)^2},$$

and

$$s(n) = u_{n-1} + u_{n+1} - u_{n-}u_{bn+1}.$$

By corollary 2.3 and theorem 2.4, we know $s(n) < 1$ holds for $n \geqslant 1207$.

Proof of theorem 2.1. Since for $222 \leqslant n \leqslant 1207$, (4.1) can be directly checked, to prove this theorem, we just only prove that (4.1) holds for $n \geqslant 1207$. Let

$$F(t) = s(n)t^2 - 2t + 1. \tag{4.2}$$

We have now to prove that for $n \geqslant 1207$,

$$F(u_n) > 0. \tag{4.3}$$

It is easily seen that the equation $F(t) = 0$ has two solutions:

$$t_1 = \frac{1-\sqrt{1-s(n)}}{s(n)}, \quad t_2 = \frac{1+\sqrt{1-s(n)}}{s(n)},$$

and thus $F(t)$ is positive when $t < t_1$ or $t > t_2$. To verify (4.3), we claim that for $n \geqslant 1207$,

$$u(n) < t_1 = \frac{1-\sqrt{1-s(n)}}{s(n)}. \tag{4.4}$$

According to theorem 2.2, we know that for $n \geqslant 1207$,

$$u(n) < g(n), \tag{4.5}$$

and from theorem 3.1, we see that for $n \geqslant 200$,

$$g(n) < \varphi(s(n)) = \frac{1-\sqrt{1-s(n)}}{s(n)}. \tag{4.6}$$

Hence, the claim is verified by combining (4.5) and (4.6). The proof is completed. □

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