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Inequalities for the overpartition function

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Abstract
Let $\overline{p}(n)$ denote the overpartition function. Engel showed that for $n \geq 2$, $\overline{p}(n)$ satisfy the Turán inequalities, that is, $\overline{p}(n)^2 - \overline{p}(n-1)\overline{p}(n+1) > 0$ for $n \geq 2$. In this paper, we prove several inequalities for $\overline{p}(n)$. Moreover, motivated by the work of Chen, Jia and Wang, we find that the higher order Turán inequalities of $\overline{p}(n)$ can also be determined.

Keywords Overpartition function · Rademacher-type series · Log-concavity · Higher order Turán inequalities

Mathematics Subject Classification 05A20 · 11P82 · 11P99

1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n. Recall that a sequence $\{a_i\}_{0 \leq i \leq n}$ is said to satisfy the Turán inequalities if

$$a_i^2 - a_{i+1}a_{i-1} \geq 0, \quad 1 \leq i \leq n.$$

In particular, a sequence satisfying the Turán inequalities can also be called log-concave. DeSalvo and Pak [9] showed that $p(n)$ is log-concave for all $n \geq 25$. They

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also proved two conjectures given by Chen [3],

$$\frac{p(n-1)}{p(n)}\left(1+\frac{1}{n}\right) > \frac{p(n)}{p(n+1)} \quad \text{for} \quad n > 1,$$

and

$$p(n)^2 - p(n-m)p(n+m) \geq 0 \quad \text{for} \quad n > m > 1.$$

Since then, the inequalities between the partition functions have been extensively studied. For example, Chen et al. [5] proved a sharper inequality

$$\frac{p(n-1)}{p(n)}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{p(n)}{p(n+1)}$$

holds for $n \geq 45$, which was conjectured by Desalvo and Pak [9]. Bessenrodt and Ono [2] obtained that

$$p(a)p(b) \geq p(a+b)$$

holds for $a, b > 1$ and $a+b > 8$. Based on this inequality, they extended the partition function multiplicatively to a function on partitions and showed that it has a unique maximum at an explicit partition for any $n \neq 7$. Recently, Dawsey and Masri [8] gave an effective asymptotic formula of the Andrews spt-function due to the algebraic formula [1] for the spt-function. According to this asymptotic formula, they proved some inequalities on the spt-function conjectured by Chen [4].

The similar inequalities can also be satisfied by the overpartition function. Recall an overpartition [7] of a nonnegative integer n is a partition of n where the first occurrence of each distinct part may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of n. Zukermann [19] gave a formula for the overpartition function, which is indeed a Rademacher-type convergent series,

$$\overline{p}(n) = \frac{1}{2\pi}\sum_{\substack{k=1\\2\nmid k}}^{\infty}\sqrt{k}\sum_{\substack{h=0\\(h,k)=1}}^{k-1}\frac{\omega(h,k)^2}{\omega(2h,k)}e^{-\frac{2\pi i n h}{k}}\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{\sinh\frac{\pi\sqrt{n}}{k}}{\sqrt{n}}\right), \tag{1.1}$$

where

$$\omega(h,k) = \exp\left(\pi i\sum_{r=1}^{k-1}\frac{r}{k}\left(\frac{hr}{k}-\left\lfloor\frac{hr}{k}\right\rfloor-\frac{1}{2}\right)\right)$$

for positive integers h and k. Let $\mu = \mu(n) = \pi\sqrt{n}$. From this Rademacher-type series (1.1), Engel [11] provided an error term for the overpartition function

$$\overline{p}(n) = \frac{1}{2\pi}\sum_{\substack{k=1\\ 2\nmid k}}^{N}\sqrt{k}\sum_{\substack{h=0\\ (h,k)=1}}^{k-1}\frac{\omega(h,k)^2}{\omega(2h,k)}e^{-\frac{2\pi inh}{k}}\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{\sinh\frac{\mu}{k}}{\sqrt{n}}\right) + R_2(n,N),$$

where

$$|R_2(n,N)| \le \frac{N^{\frac{5}{2}}}{n\mu}\sinh\left(\frac{\mu}{N}\right). \tag{1.2}$$

In particular, when $N = 2$, we have

$$\overline{p}(n) = \frac{1}{8n}\left[\left(1+\frac{1}{\mu}\right)e^{-\mu} + \left(1-\frac{1}{\mu}\right)e^{\mu}\right] + R_2(n,2), \tag{1.3}$$

where

$$|R_2(n,2)| \le \frac{2^{\frac{5}{2}}}{n\mu}\sinh\left(\frac{\mu}{2}\right). \tag{1.4}$$

Moreover, using this asymptotic formula (1.3), Engel [11] proved that $\overline{p}(n)$ is log-concave for $n \ge 2$, that is,

$$\overline{p}(n)^2 > \overline{p}(n-1)\overline{p}(n+1). \tag{1.5}$$

Let Δ be the difference operator as given by $\Delta f(n) = f(n+1) - f(n)$. Recently, Wang et al. [18] showed that for any given $r \ge 1$, there exists a positive number $n(r)$ such that $(-1)^{r-1}\Delta^r \log\overline{p}(n) > 0$ for $n > n(r)$. Moreover, they gave an upper bound for $(-1)^{r-1}\Delta^r \log\overline{p}(n)$. More precisely, for all $r \ge 1$, there exists a positive integer $n(r)$ such that for $n > n(r)$,

$$(-1)^{r-1}\Delta^r \log\overline{p}(n) < \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1}\frac{1}{n^{r-1/2}},$$

where $(x)_n := x\cdot(x+1)\cdots(x+n-1)$. From the proof of [18, Theorem 4.1], we can obtain a slight modification of this result as follows:

$$(-1)^{r-1}\Delta^r \log\overline{p}(n-1) < \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1}\frac{1}{n^{r-1/2}}.$$

In particular, when $r = 2$, we have

$$\frac{\overline{p}(n-1)}{\overline{p}(n)}\left(1+\frac{\pi}{4n^{3/2}}\right) \ge \frac{\overline{p}(n)}{\overline{p}(n+1)} \quad \text{for} \quad n \ge 2. \tag{1.6}$$

In this paper, we prove some inequalities for the overpartition function. One of main results of this paper is the following theorem analogous to these inequalities for the partition function obtained by DeSalvo and Pak [9] and Bessenrodt and Ono [2].

Theorem 1.1 (1) *For all $n > m > 1$, we have*

$$\overline{p}(n)^2 - \overline{p}(n-m)\overline{p}(n+m) \geq 0, \tag{1.7}$$

with equality holding only for $(n, m) = (2, 1)$.

(2) *If a, b are integers with $a, b > 1$, then*

$$\overline{p}(a)\overline{p}(b) > \overline{p}(a+b). \tag{1.8}$$

To state the second result, we first introduce some definitions. Given a function $\gamma : \mathbb{N} \to \mathbb{R}$ and positive integers d and n, the associated Jensen polynomial of degree d and shift n is defined by

$$J_\gamma^{d,n}(n) := \sum_{j=0}^{d} \binom{d}{j} \gamma_{n+j} X^j.$$

If all of zeros of a polynomial are real, then this polynomial is said to be hyperbolic. A real entire function

$$\psi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!}$$

is said to be in the Laguerre–Pólya class if it can be represented in the form

$$\psi(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} (1 + x/x_k) e^{-x/x_k},$$

where c, β, x_k are real numbers, $\alpha \geq 0$, n is a nonnegative integer and $\sum x_k^{-2} < \infty$. Jensen [14] showed that $\psi(x)$ belongs to the Laguerre–Pólya class if and only if all of the associated Jensen polynomials $J_\gamma^{d,0}$ are hyperbolic. Pólya [16] proved that the Riemann Hypothesis is equivalent to the hyperbolicity of all Jensen polynomials associated to Riemann's ξ-function.

The Turán inequalities and the higher order Turán inequalities are related to the Lagurre–Pólya class of real entire functions. From the work of Pólya and Schur [17] we see that the Maclaurin coefficients of $\psi(x)$ in the Lagurre–Pólya class satisfy the Turán inequalities

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0$$

for $k \geq 1$. Due to the result of Dimitrov [10], we know that the Macalurin coefficients of $\psi(x)$ in the Lagurre–Pólya class satisfy the higher order Turán inequalities

$$4(\gamma_k^2 - \gamma_{k-1}\gamma_{k+1})(\gamma_{k+1}^2 - \gamma_k\gamma_{k+2}) - (\gamma_k\gamma_{k+1} - \gamma_{k-1}\gamma_{k+2})^2 \geq 0$$

for $k \geq 1$.

Clearly, from the results of Desalvo and Pak [9], Engel [11] and Dawsey and Masri [8], we see that the partition function, the overpartition function and the spt-function all satisfy the Turán inequalities. Moreover, Chen et al. [6] showed that the partition function satisfy the higher order Turán inequalities. In this paper, we confirm the overpartition function also satisfy the higher order Turán inequalities.

Theorem 1.2 *Let*

$$u_n = \frac{\overline{p}(n-1)\overline{p}(n+1)}{\overline{p}(n)^2}.$$

For $n \geq 16$,

$$4(1-u_n)(1-u_{n+1}) - (1-u_n u_{n+1})^2 > 0.$$

Remark 1.3 Recently, Griffin et al. [12] proved that Jensen polynomials for weakly holomorphic modular forms on $SL_2(\mathbb{Z})$ with real coefficients and a pole at $i\infty$ are eventually hyperbolic. This work proved Chen et al.'s conjecture [6] that the Jensen polynomials associated to the partition function $p(n)$ are eventually hyperbolic as a special case. In other words, for each $d \geq 1$ there exists some $N(d)$ such that for all $n \geq N(d)$, the polynomial $J_p^{d,n}(x)$ is hyperbolic. Larson and Wagner [15] computed the values of the minimal $N(d)$ for $d = 3, 4, 5$ and gave an upper bound of the minimal $N(d)$ for each $d \geq 1$. Moreover, the work of Griffin et al. [12] can also be used to prove that the Jensen polynomials associated to the overpartition function $\overline{p}(n)$ are eventually hyperbolic. In this paper, we give an explicit bound for the Jensen polynomial $J_{\overline{p}}^{3,n}(x)$, that is, for all $n \geq 16$, $J_{\overline{p}}^{3,n}(x)$ is hyperbolic.

2 Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. To prove the part (1) of Theorem 1.1, we need the following lemma, which is looser but simpler than (1.3) and (1.4).

Lemma 2.1 *For all* $n \geq 1$, *we have*

$$\overline{p}(n) = \alpha(n)e^{\mu} + E_{\overline{p}}(n), \tag{2.1}$$

where

$$\alpha(n) = \frac{1}{8n}\left(1 - \frac{1}{\mu}\right),$$

and

$$|E_{\overline{p}}(n)| \leq \frac{e^{\mu/2}}{n^{3/2}}.$$

Proof By (1.3) and (1.4), we obtain that

$$|E_{\overline{p}}(n)| \leq \frac{e^{-\mu}}{8n}\left(1+\frac{1}{\mu}\right)+\frac{2^{\frac{5}{2}}}{n\mu}\sinh\left(\frac{\mu}{2}\right). \tag{2.2}$$

Define

$$g(n)=\frac{e^{-\mu}}{8n}\left(1+\frac{1}{\mu}\right).$$

Clearly, $g(n)$ is monotonically decreasing for $n>0$. For $n \geq 1$,

$$g(n)<g(1)=\frac{(1+\pi)e^{-\pi}}{8\pi}<0.0072.$$

Making use of the fact that

$$\sinh(x)<e^{x}/2, \quad \text{for} \quad x>0,$$

we see that

$$|E_{\overline{p}}(n)| \leq 0.0072+\frac{2^{\frac{3}{2}}e^{\frac{\mu}{2}}}{n\mu}. \tag{2.3}$$

Letting

$$f(x)=\frac{e^{\frac{\pi\sqrt{x}}{2}}}{x^{3/2}}\left(1-\frac{2^{\frac{3}{2}}}{\pi}\right),$$

we find that for $x>1$, the minimum of $f(x)$ is at $x=36/\pi^2 \approx 3.65$, and $f\left(36/\pi^2\right)>0.287$, hence we have

$$\frac{e^{\frac{\mu}{2}}}{n^{3/2}}-\frac{2^{\frac{3}{2}}e^{\frac{\mu}{2}}}{n\mu}>0.0072 \quad \text{for} \quad n \geq 1. \tag{2.4}$$

The proof follows from (2.3) and (2.4). □

Using the estimate of the overpartition function in Lemma 2.1, we are ready to give a proof of the first part of Theorem 1.1 which is analogous to the proof of the corresponding theorem for $p(n)$ in [9].

Proof of Theorem 1.1 (1) We already know that the sequence $\overline{p}(n)$ satisfies (1.5). It is known that log-concavity implies strong log-concavity, that is

$$\overline{p}(k)\overline{p}(\ell) \leq \overline{p}(\ell - i)\overline{p}(k + i),$$

for all $0 \leq k \leq \ell$ and $0 \leq i \leq \ell - k$. In particular, we take $k = n - m$, $\ell = n + m$ and $i = m$ in the above inequality to obtain

$$\overline{p}(n)^2 - \overline{p}(n - m)\overline{p}(n + m) > 0$$

for all $n > m > 1$ with $n - m > 1$.

Now we consider the case $n > m > 1$ with $n = m + 1$. It suffices to show that

$$\overline{p}(m + 1)^2 > \overline{p}(1)\overline{p}(2m + 1) \tag{2.5}$$

for all $m \geq 2$. Taking logarithms in the inequality above, we see that it is equivalent to prove that

$$2 \log \overline{p}(m + 1) - \log \overline{p}(1) - \log \overline{p}(2m + 1) > 0 \tag{2.6}$$

for all $m \geq 2$. Moreover, it follows from Lemma 2.1 that for $m \geq 5$,

$$\frac{1}{8m}\left(1 - \frac{2}{\mu(m)}\right)e^{\mu(m)} < \overline{p}(m) < \frac{1}{8m}\left(1 + \frac{1}{\mu(m)}\right)e^{\mu(m)}. \tag{2.7}$$

Combining (2.7) with (2.6), we deduce that

$$\begin{aligned}&-2\log(8m + 8) + 2\log\left(1 - \frac{2}{\mu(m + 1)}\right) + 2\mu(m + 1) - \log 2 + \log(16m + 8)\\ &\quad - \log\left(1 + \frac{1}{\mu(2m + 1)}\right) - \mu(2m + 1) > 0\end{aligned}$$

for all $m \geq 5$. It is checked directly that (2.5) holds for the cases $m = 2$, 3 and 4. □

Next we will prove the second part of Theorem 1.1 due to Engel's bound

$$\overline{p}(n) = \frac{1}{8n}\left[\left(1 + \frac{1}{\mu}\right)e^{-\mu} + \left(1 - \frac{1}{\mu}\right)e^{\mu}\right] + R_2(n, 2),$$

where

$$|R_2(n, 2)| \leq \frac{2^{\frac{5}{2}}}{n\mu}\sinh\left(\frac{\mu}{2}\right).$$

It closely follows the proofs of the corresponding theorems for $p(n)$ and $\mathrm{spt}(n)$ in [2] and [8].

Proof of Theorem 1.1 (2) Engle [11, p. 234] showed that

$$|R_2(n,N)| \leq \sum_{m=1}^{\infty} \frac{4m}{4m-3} \frac{\left(\frac{\mu(n)}{N}\right)^{2m}}{(2m+1)!} \frac{N^{3/2}}{4n}.$$

From this inequality we can modify the bound of $R_2(n, N)$ slightly,

$$\begin{aligned} |R_2(n,N)| &\leq \frac{N^{3/2}}{n} \sum_{m=1}^{\infty} \frac{\left(\frac{\mu(n)}{N}\right)^{2m}}{(2m+1)!} \\ &= \frac{N^{5/2}}{n\mu} \left(\sinh\left(\frac{\mu}{N}\right) - \frac{\mu}{N}\right). \end{aligned}$$

For $N = 2$, we have

$$|R_2(n,2)| \leq \frac{2^{5/2}}{n\mu} \left(\sinh\left(\frac{\mu}{2}\right) - \frac{\mu}{2}\right) \leq \frac{2^{\frac{5}{2}}}{n\mu} \left[\sinh\left(\frac{\mu}{2}\right) - 1\right]. \tag{2.8}$$

Thanks to this error bound (2.8), we obtain the upper bound of $\overline{p}(n)$

$$\overline{p}(n) < \frac{e^{\mu}}{8n} \left(1 + \frac{1}{n}\right) \quad \text{for} \quad n \geq 1. \tag{2.9}$$

On the other hand, it follows from (2.7) that the lower bound of $\overline{p}(n)$ is

$$\overline{p}(n) > \frac{e^{\mu}}{8n} \left(1 - \frac{1}{\sqrt{n}}\right) \quad \text{for} \quad n \geq 1.$$

We may assume $1 < a \leq b$, for convenience, we let $b = \lambda a$, where $\lambda \geq 1$. These inequalities immediately give

$$\begin{aligned} \overline{p}(a)\overline{p}(\lambda a) &> \frac{e^{\mu(a)+\mu(\lambda a)}}{64\lambda a^2} \left(1 - \frac{1}{\sqrt{a}}\right) \left(1 - \frac{1}{\sqrt{\lambda a}}\right), \\ \overline{p}(a + \lambda a) &< \frac{e^{\mu(a+\lambda a)}}{8a(\lambda+1)} \left(1 + \frac{1}{a + \lambda a}\right). \end{aligned}$$

For all but finitely many cases, it suffices to find conditions on $a > 1$ and $\lambda \geq 1$ for which

$$\frac{e^{\mu(a)+\mu(\lambda a)}}{64\lambda a^2} \left(1 - \frac{1}{\sqrt{a}}\right) \left(1 - \frac{1}{\sqrt{\lambda a}}\right) > \frac{e^{\mu(a+\lambda a)}}{8a(\lambda+1)} \left(1 + \frac{1}{a + \lambda a}\right).$$

Since $\lambda \geq 1$, we have that $\lambda/(\lambda + 1) \geq 1/2$, hence it suffices to consider when

$$e^{\mu(a)+\mu(\lambda a)-\mu(a+\lambda a)} > 4aS_a(\lambda),$$

Table 1 Values of λ_a

a	λ_a
2	7.578...
3	2.566...
4	1.550...
5	1.117...

where

$$S_a(\lambda) = \frac{1 + \frac{1}{a+\lambda a}}{\left(1 - \frac{1}{\sqrt{a}}\right)\left(1 - \frac{1}{\sqrt{\lambda a}}\right)}. \tag{2.10}$$

By taking the logarithm, we obtain the inequality

$$T_a(\lambda) > \log(4a) + \log(S_a(\lambda)), \tag{2.11}$$

where

$$T_a(\lambda) = \pi\left(\sqrt{a} + \sqrt{\lambda a} - \sqrt{a + \lambda a}\right). \tag{2.12}$$

We consider (2.10) and (2.12) as functions in $\lambda \geq 1$ and fixed $a > 1$. By simple calculations, we find that $S_a(\lambda)$ is decreasing in $\lambda \geq 1$, while $T_a(\lambda)$ is increasing in $\lambda \geq 1$. Therefore, (2.11) becomes

$$T_a(\lambda) \geq T_a(1) > \log(4a) + \log(S_a(1)) \geq \log(4a) + \log(S_a(\lambda)).$$

By evaluating $T_a(1)$ and $S_a(1)$ directly, one easily finds that (2.11) holds whenever $a \geq 6$. To complete the proof, assume that $2 \leq a \leq 5$. We then directly calculate the real number λ_a for which

$$T_a(\lambda_a) = \log(4a) + \log(S_a(\lambda_a)).$$

By the discussion above, if $b = \lambda a \geq a$ is an integer for which $\lambda > \lambda_a$, then (2.11) holds, which in turn gives the theorem in these cases. Table 1 gives the numerical calculations for these λ_a. Only finitely many cases remain, namely the pairs of integers where $2 \leq a \leq 5$ and $1 \leq b/a \leq \lambda_a$. We compute $\overline{p}(a)$, $\overline{p}(b)$ and $\overline{p}(a+b)$ for these cases to complete the proof. □

3 Proof of Theorem 1.2

In this section, we employ the method of Chen et al. [6], which is used to prove the third order Turán inequality for the partition function, to prove the third order Turán inequality for the overpartition function

$$4(1-u_n)(1-u_{n+1})-(1-u_nu_{n+1})^2>0 \quad \text{for} \quad n\geq 16.$$

To this end, we first bound the ratio $u_n=\overline{p}(n-1)\overline{p}(n+1)/\overline{p}(n)^2$. Then we build some inequalities among $\mu=\mu(n)=\pi\sqrt{n}$ and the lower bound $f(n)$ and the upper bound $g(n)$ for u_n. Finally, the distribution of the roots of the polynomial

$$F(t)=4(1-u_n)(1-t)-(1-u_nt)^2$$

gives us the chance to prove the third order Turán inequality for the overpartition function.

Next we find an effective bound for the overpartition function $\overline{p}(n)$ and then give the upper and lower bounds of u_n,

Theorem 3.1 *For $n\geq 55$,*

$$f(n)<u_n<g(n), \tag{3.1}$$

where

$$x=\mu(n-1),\quad y=\mu=\mu(n),\quad z=\mu(n+1),\quad w=\mu(n+2),$$

and

$$f(n)=e^{x-2y+z}\frac{y^{14}(x^5-x^4-1)(z^5-z^4-1)}{x^7z^7(y^5-y^4+1)^2}, \tag{3.2}$$

$$g(n)=e^{x-2y+z}\frac{y^{14}(x^5-x^4+1)(z^5-z^4+1)}{x^7z^7(y^5-y^4-1)^2}. \tag{3.3}$$

Proof Let

$$B_1(n)=\frac{e^{\mu}}{8n}\left(1-\frac{1}{\mu}-\frac{1}{\mu^5}\right),$$

$$B_2(n)=\frac{e^{\mu}}{8n}\left(1-\frac{1}{\mu}+\frac{1}{\mu^5}\right).$$

We first claim that the following bounds for the overpartition function $\overline{p}(n)$ holds,

$$B_1(n)<\overline{p}(n)<B_2(n) \quad \text{for} \quad n\geq 55. \tag{3.4}$$

Set

$$\widetilde{T}(n)=\left(1+\frac{1}{\mu}\right)e^{-2\mu}+\frac{8n}{e^{\mu}}R_2(n,2).$$

So we can rewrite (1.3) as

$$\overline{p}(n) = \frac{e^{\mu}}{8n}\left(1 - \frac{1}{\mu} + \widetilde{T}(n)\right), \tag{3.5}$$

where

$$|R_2(n,2)| \le \frac{2^{\frac{5}{2}}}{n\mu}\sinh\left(\frac{\mu}{2}\right) \le \frac{2^{\frac{3}{2}}e^{\frac{\mu}{2}}}{n\mu}.$$

Obviously, for $n \ge 1$,

$$0 < \frac{1}{\mu} < \frac{1}{2},$$

we have

$$\left(1 + \frac{1}{\mu}\right)e^{-2\mu} < 2e^{-2\mu} < 2e^{-\frac{1}{2}\mu}.$$

As for the last term in $\widetilde{T}(n)$,

$$\left|\frac{8n}{e^{\mu}}R_2(n,2)\right| < \left|2^{\frac{9}{2}}\frac{e^{-\frac{1}{2}\mu}}{\mu}\right| < 8e^{-\frac{1}{2}\mu}.$$

Thus

$$|\widetilde{T}(n)| < 10e^{-\frac{1}{2}\mu}. \tag{3.6}$$

Next we aim to prove that for $n \ge 254$,

$$10e^{-\frac{1}{2}\mu} < \frac{1}{\mu^5}, \tag{3.7}$$

which can be recast as

$$\frac{e^{\mu/10}}{\mu/10} > 10 \cdot \sqrt[5]{10}.$$

Let $F(t) = e^t/t$. Since $F'(t) = e^t(t-1)/t^2 > 0$ for $t > 1$, $F(t)$ is increasing for $t > 1$. Observe that for $n > 253$, $\mu/10 > 5$. Thus,

$$F\left(\frac{\mu}{10}\right) = \frac{e^{\mu/10}}{\mu/10} > F(5) = \frac{e^5}{5} > 10\sqrt[5]{10}.$$

So (3.7) holds for $n \geq 254$. Thus, combining (3.6) and (3.7), we get that for $n \geq 254$,

$$-\frac{1}{\mu^5} < \widetilde{T}(n) < \frac{1}{\mu^5}. \tag{3.8}$$

Substituting (3.8) into (3.5), we see that (3.4) holds for $n \geq 254$. It is routine to check that (3.4) is true for $55 \leq n \leq 253$, and hence the claim (3.4) can be verified.

Since $B_1(n)$ and $B_2(n)$ are all positive for $n \geq 1$, using the bounds for $\overline{p}(n)$ in (3.4), we find that for $n \geq 55$,

$$\frac{B_1(n-1)B_1(n+1)}{B_2(n)^2} < \frac{\overline{p}(n-1)\overline{p}(n+1)}{\overline{p}(n)^2} < \frac{B_2(n-1)B_2(n+1)}{B_1(n)^2},$$

and which completes the proof. □

Now we will build an inequality between $f(n)$ and $g(n+1)$.

Theorem 3.2 *For $n \geq 2$,*

$$g(n+1) < f(n) + \frac{1000}{\mu(n-1)^5}. \tag{3.9}$$

Proof Recall that

$$\mu(n) = \pi\sqrt{n},$$

and

$$x = \mu(n-1), \quad y = \mu(n), \quad z = \mu(n+1), \quad w = \mu(n+2).$$

Let

$$\alpha(t) = t^5 - t^4 + 1, \quad \beta(t) = t^5 - t^4 - 1.$$

By (3.2) and (3.3), we see that

$$f(n)x^5 - g(n+1)x^5 + 1000 = \frac{-e^{w+y-2z}t_1 + e^{z+x-2y}t_2 + 1000t_3}{t_3},$$

where

$$t_1 = x^7 z^{21}\alpha(y)^3\alpha(w), \tag{3.10}$$

$$t_2 = y^{21} w^7 \beta(x)\beta(z)^3, \tag{3.11}$$

$$t_3 = x^2 y^7 z^7 w^7 \alpha(y)^2 \beta(z)^2. \tag{3.12}$$

Since $t_3 > 0$ for $n \geq 2$, (3.9) is equivalent to

$$-e^{w+y-2z}t_1 + e^{z+x-2y}t_2 + 1000t_3 > 0$$

for $n \geq 2$. To do this, we need to estimate $t_1, t_2, t_3, e^{w+y-2z}$ and e^{x-2y+z} in terms of x. Note that for $n \geq 2$,

$$y = \sqrt{x^2+\pi^2}, \quad z = \sqrt{x^2+2\pi^2}, \quad w = \sqrt{x^2+3\pi^2}.$$

Then for $x > 1$, we have the following expansions:

$$y = x + \frac{\pi^2}{2x} - \frac{\pi^4}{8x^3} + \frac{\pi^6}{16x^5} - \frac{5\pi^8}{128x^7} + \frac{7\pi^{10}}{256x^9} - \frac{21\pi^{12}}{1024x^{11}} + O\left(\frac{1}{x^{12}}\right),$$
$$z = x + \frac{\pi^2}{x} - \frac{\pi^4}{2x^3} + \frac{\pi^6}{2x^5} - \frac{5\pi^8}{8x^7} + \frac{7\pi^{10}}{8x^9} - \frac{21\pi^{12}}{16x^{11}} + O\left(\frac{1}{x^{12}}\right),$$
$$w = x + \frac{3\pi^2}{2x} - \frac{9\pi^4}{8x^3} + \frac{27\pi^6}{16x^5} - \frac{405\pi^8}{128x^7} + \frac{1701\pi^{10}}{256x^9} - \frac{15309\pi^{12}}{1024x^{11}} + O\left(\frac{1}{x^{12}}\right).$$

It is easy to see that for $x > 1$,

$$y_1 < y < y_2, \tag{3.13}$$
$$z_1 < z < z_2, \tag{3.14}$$
$$w_1 < w < w_2, \tag{3.15}$$

where

$$y_1 = x + \frac{\pi^2}{2x} - \frac{\pi^4}{8x^3} + \frac{\pi^6}{16x^5} - \frac{5\pi^8}{128x^7} + \frac{7\pi^{10}}{256x^9} - \frac{21\pi^{12}}{1024x^{11}},$$
$$y_2 = x + \frac{\pi^2}{2x} - \frac{\pi^4}{8x^3} + \frac{\pi^6}{16x^5} - \frac{5\pi^8}{128x^7} + \frac{7\pi^{10}}{256x^9},$$
$$z_1 = x + \frac{\pi^2}{x} - \frac{\pi^4}{2x^3} + \frac{\pi^6}{2x^5} - \frac{5\pi^8}{8x^7} + \frac{7\pi^{10}}{8x^9} - \frac{21\pi^{12}}{16x^{11}},$$
$$z_2 = x + \frac{\pi^2}{x} - \frac{\pi^4}{2x^3} + \frac{\pi^6}{2x^5} - \frac{5\pi^8}{8x^7} + \frac{7\pi^{10}}{8x^9},$$
$$w_1 = x + \frac{3\pi^2}{2x} - \frac{9\pi^4}{8x^3} + \frac{27\pi^6}{16x^5} - \frac{405\pi^8}{128x^7} + \frac{1701\pi^{10}}{256x^9} - \frac{15309\pi^{12}}{1024x^{11}},$$
$$w_2 = x + \frac{3\pi^2}{2x} - \frac{9\pi^4}{8x^3} + \frac{27\pi^6}{16x^5} - \frac{405\pi^8}{128x^7} + \frac{1701\pi^{10}}{256x^9}.$$

Next we make use of these bounds of y, z and w in (3.13), (3.14) and (3.15) to estimate $t_1, t_2, t_3, e^{w+y-2z}$ and e^{x-2y+z} in terms of x.

First, we give estimates for t_1, t_2 and t_3. We use (3.15) to derive that for $x > 1$,

$$w_1 w^4 < w^5 < w_2 w^4.$$

Let

$$\eta_1 = w_2 w^4 - w^4 + 1,$$

so that for $x > 1$,

$$\alpha(w) < \eta_1. \tag{3.16}$$

Similarly, set

$$\begin{aligned}
\eta_2 &= y_2 y^{14} - 3y^{14} + 3y_2 y^{12} - y^{12} + 3y^{10} - 6y_1 y^8 + 3y^8 + 3y_2 y^4 - 3y^4 + 1,\\
\eta_3 &= z_1 z^{14} - 3z^{14} + 3z_1 z^{12} - z^{12} - 3z^{10} + 6z_1 z^8 - 3z^8 + 3z_1 z^4 - 3z^4 - 1,\\
\eta_4 &= y^{10} - 2y_2 y^8 + y^8 + 2y_1 y^4 - 2y^4 + 1,\\
\eta_5 &= z^{10} - 2z_2 z^8 + z^8 - 2z_2 z^4 + 2z^4 + 1.
\end{aligned}$$

Then we have for $x > 1$,

$$\alpha(y)^3 < \eta_2, \quad \beta(z)^3 > \eta_3, \quad \alpha(y)^2 > \eta_4, \quad \beta(z)^2 > \eta_5. \tag{3.17}$$

Together the relations in (3.16) and (3.17), we find that for $x > 1$,

$$t_1 = x^7 z^{21} \alpha(y)^3 \alpha(w) < x^7 z_2 z^{20} \eta_1 \eta_2, \tag{3.18}$$

$$t_2 = y^{21} w^7 (x^5 - x^4 - 1)\beta(z)^3 > y_1 y^{20} w_1 w^6 (x^5 - x^4 - 1)\eta_3, \tag{3.19}$$

$$t_3 = x^2 y^7 z^7 w^7 \alpha(y)^2 \beta(z)^2 > x^2 y_1 y^6 z_1 z^6 w_1 w^6 \eta_4 \eta_5. \tag{3.20}$$

We continue to estimate e^{w+y-2z} and e^{z+x-2y}. Applying (3.13), (3.14) and (3.15) to $w + y - 2z$, we see that for $x > 1$,

$$w + y - 2z < w_2 + y_2 - 2z_1, \tag{3.21}$$

which implies that

$$e^{w+y-2z} < e^{w_2+y_2-2z_1}. \tag{3.22}$$

We define

$$\Phi(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \frac{t^6}{720}, \tag{3.23}$$

so as to give a feasible upper bound for e^{w+y-2z}, Then we have that for $t < 0$,

$$e^t < \Phi(t). \tag{3.24}$$

Since $\pi^4(16x^8 - 48\pi^2 x^6) > 0$ and $125\pi^4 x^4 - 315\pi^6 x^2 - 168\pi^8 > 0$ both hold for $x \geq 6$,

$$w_2 + y_2 - 2z_1 = -\frac{\pi^4(16x^8 - 48\pi^2 x^6 + 125\pi^4 x^4 - 315\pi^6 x^2 - 168\pi^8)}{64x^{11}} < 0$$

holds for $x \geq 6$. Thus, we deduce that for $x \geq 6$

$$e^{w_2 + y_2 - 2z_1} < \Phi(w_2 + y_2 - 2z_1). \tag{3.25}$$

Then it follows from (3.22) and (3.25) that for $x \geq 6$,

$$e^{w+y-2z} < \Phi(w_2 + y_2 - 2z_1). \tag{3.26}$$

Similarly, applying (3.13), (3.14) and (3.15) to $z + x - 2y$, we find that for $x > 1$,

$$z_1 + x - 2y_2 < z + x - 2y \tag{3.27}$$

so that

$$e^{z_1 + x - 2y_2} < e^{z+x-2y}. \tag{3.28}$$

Define

$$\phi(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \frac{t^6}{720} + \frac{t^7}{5040}. \tag{3.29}$$

It can be easily verified that for $t < 0$, $\phi(t) < e^t$. Since

$$\begin{aligned} z + x - 2y &= \sqrt{x^2 + 2\pi^2} + x - 2\sqrt{x^2 + \pi^2} \\ &= -\frac{\left(\sqrt{x^2 + 2\pi^2} - x\right)^2}{\sqrt{x^2 + 2\pi^2} + x + 2\sqrt{x^2 + \pi^2}} < 0 \end{aligned}$$

for $x \geq 5$, we deduce that for $x \geq 5$,

$$z_1 + x - 2y_2 < 0.$$

Thus, we get that for $x \geq 5$,

$$\phi(z_1 + x - 2y_2) < e^{z_1 + x - 2y_2}. \tag{3.30}$$

Combining (3.28) and (3.30) yields that for $x \geq 5$,

$$e^{z+x-2y} > \phi(z_1 + x - 2y_2). \tag{3.31}$$

Using the above bounds for t_1, t_2, t_3, e^{w+y-2z} and e^{x-2y+z}, we obtain that for $x \geq 6$,

$$\begin{aligned}
&-e^{w+y-2z}t_1 + e^{z+x-2y}t_2 + 1000t_3 \\
&\quad > -\Phi(w_2 + y_2 - 2z_1)x^7 z_2 z^{20}\eta_1\eta_2 \\
&\quad\quad + \phi(z_1 + x - 2y_2)y_1 w_1 y^{20} w^6(x^5 - x^4 - 1)\eta_3 \\
&\quad\quad + 1000x^2 y_1 z_1 w_1 y^6 z^6 w^6 \eta_4\eta_5.
\end{aligned}$$

It remains to verify that for $x \geq 5$,

$$\begin{aligned}
&-\Phi(w_2 + y_2 - 2z_1)x^7 z_2 z^{20}\eta_1\eta_2 + \phi(z_1 + x - 2y_2)y_1 w_1 y^{20} w^6(x^5 - x^4 - 1)\eta_3 \\
&\quad + 1000x^2 y_1 z_1 w_1 y^6 z^6 w^6 \eta_4\eta_5 > 0.
\end{aligned}$$

Replacing y, z and w by $\sqrt{x^2+\pi^2}$, $\sqrt{x^2+2\pi^2}$ and $\sqrt{x^2+3\pi^2}$, respectively, we see that the left-hand side of above inequality can be expressed as $H(x)/G(x)$, where

$$H(x) = \sum_{k=0}^{153} a_k x^k$$

and

$$G(x) = 476014541473260237540556 80x^{110}.$$

Here we just list the last few values of

$$\begin{aligned}
a_{151} &= 148754544210393824231424 0 \\
&\quad \times \left(191232 + 1143744\pi^2 - 388\pi^6 - 387\pi^8\right), \\
a_{152} &= 166605089515641083139194880\left(-1136 + \pi^6\right), \\
a_{153} &= 59501817684157529692569 60\left(7936 - 3\pi^6\right),
\end{aligned}$$

which a_{151} and a_{153} are positive, but a_{152} is negative.

Because $G(x)$ is always positive for all positive x, it suffices to prove that $H(x) > 0$. It is clear that $x \geq 2$ for $n \geq 2$ and hence

$$H(x) \geq \sum_{k=0}^{150} -|a_k|x^k + a_{151}x^{151} + a_{152}x^{152} + a_{153}x^{153}.$$

Moreover, numerical evidence indicates that for any $0 \leq k \leq 150$,

$$-|a_k|x^k > -a_{151}x^{151}$$

holds for $x \geq 14$. It follows that for $x \geq 14$,

$$\sum_{k=0}^{150} -|a_k|x^k + a_{152}x^{152} + a_{153}x^{153} > -151a_{151}x^{151} + a_{151}x^{151} + a_{152}x^{152} + a_{153}x^{153},$$

which yields that

$$H(x) > \left(-150a_{151} + a_{152}x + a_{153}x^2\right) x^{151}.$$

Thus, $H(x)$ is positive provided

$$-150a_{151} + a_{152}x + a_{153}x^2 > 0,$$

which is true if

$$x > \frac{-a_{152} + \sqrt{a_{152}^2 + 600a_{151}a_{153}}}{2a_{153}} \approx 235.402.$$

So we conclude that $H(x)$ is positive if $x \geq 236$. Therefore, for $x \geq 236$, or equivalently, for $n \geq 5615$,

$$-e^{w+y-2z}t_1 + e^{z+x-2y}t_2 + 1000t_3 > 0. \tag{3.32}$$

For $2 \leq n \leq 5614$, (3.32) can be directly verified. So we complete the proof. □

The following result is an inequality on u_n and $f(n)$ and is also an important step to prove the third Turán inequality in Theorem 1.2.

Theorem 3.3 *For* $0 < t < 1$, *let*

$$Q(t) = \frac{3t + 2\sqrt{(1-t)^3} - 2}{t^2}. \tag{3.33}$$

Then for $n \geq 92$,

$$f(n) + \frac{1000}{\mu(n-1)^5} < Q(u_n). \tag{3.34}$$

Before we give a proof of Theorem 3.3, we need the following lemma. Recall that

$$f(n) = e^{x-2y+z} \frac{y^{14}(x^5 - x^4 - 1)(z^5 - z^4 - 1)}{x^7z^7(y^5 - y^4 + 1)^2}$$

and

$$\Phi(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \frac{t^6}{720}.$$

Lemma 3.4 *For $n \geq 4$, we have*

$$f(n) < \frac{\Phi(x - 2y_1 + z_2)(x^5 - x^4 - 1)y^{14}(z_2 z^4 - z^4 - 1)}{x^7(y^{10} - 2y_2 y^8 + y^8 + 2y_1 y^4 - 2y^4 + 1)z_1 z^6} < 1, \tag{3.35}$$

where y_1, y_2, z_1 and z_2 are defined in the proof of Theorem 3.2.

Proof From (3.13) and (3.14) we see that for $x \geq 1$,

$$e^{x-2y+z} < e^{x-2y_1+z_2}, \tag{3.36}$$
$$z^5 - z^4 - 1 < z_2 z^4 - z^4 - 1, \tag{3.37}$$
$$(y^5 - y^4 + 1)^2 > y^{10} - 2y_2 y^8 + y^8 + 2y_1 y^4 - 2y^4 + 1. \tag{3.38}$$

Now we give an upper bound for $e^{x-2y_1+z_2}$. Notice that

$$x - 2y_1 + z_2 = -\frac{\pi^4\left(128x^8 - 192\pi^2 x^6 + 280\pi^4 x^4 - 420\pi^6 x^2 - 21\pi^8\right)}{512x^{11}}. \tag{3.39}$$

Moreover, it is easily verified that

$$128x^8 - 192\pi^2 x^6 > 0 \quad \text{for} \quad x \geq 4,$$

and

$$280\pi^4 x^4 - 420\pi^6 x^2 - 21\pi^8 > 0 \quad \text{for} \quad x \geq 4.$$

Therefore, $x - 2y_1 + z_2 < 0$ holds for $x \geq 4$. It follows from (3.24) that for $x \geq 4$,

$$e^{x-2y_1+z_2} < \Phi(x - 2y_1 + z_2). \tag{3.40}$$

Combining (3.36) with (3.40), we find that for $x \geq 4$,

$$e^{x-2y+z} < \Phi(x - 2y_1 + z_2). \tag{3.41}$$

Together with (3.37), (3.38) and (3.41), we see that the first inequality in (3.35) holds for $x \geq 4$, or equivalently, $n \geq 2$.

To prove the second inequality in (3.35), we define the polynomial $H(x)$ and $G(x)$ to be the numerator and denominator of

$$\frac{\Phi(x-2y_1+z_2)(x^5-x^4-1)y^{14}(z_2z^4-z^4-1)}{x^7(y^{10}-2y_2y^8+y^8+2y_1y^4-2y^4+1)z_1z^6},$$

respectively. It is easy to see that $H(x)$ and $G(x)$ are both polynomials of degree 99. For convenience, write

$$H(x)=\sum_{k=0}^{99} b_k x^k, \quad G(x)=\sum_{k=0}^{99} c_k x^k. \tag{3.42}$$

Here are the values of b_k and c_k for $94 \le k \le 99$:

$$\begin{aligned}
b_{94} &= -2^{58}\cdot 3^2\cdot 5\cdot(16+934\pi^4+21\pi^6),\\
b_{95} &= 2^{61}\cdot 3^2\cdot 5\cdot \pi^2\cdot(11+64\pi^2),\\
b_{96} &= -2^{59}\cdot 3^2\cdot 5\cdot \pi^2\cdot(92+\pi^2),\\
c_{94} &= 2^{59}\cdot 3^2\cdot 5\cdot(8-455\pi^4),\\
c_{95} &= 2^{60}\cdot 3^2\cdot 5\cdot \pi^2\cdot(22+125\pi^2),\\
c_{96} &= -2^{61}\cdot 3^2\cdot 5\cdot 23\cdot \pi^2,\\
b_{97} &= c_{97} = 2^{61}\cdot 3^2\cdot 5\cdot(1+12\pi^2),\\
b_{98} &= c_{98} = -2^{62}\cdot 3^2\cdot 5,\\
b_{99} &= c_{99} = 2^{61}\cdot 3^2\cdot 5.
\end{aligned}$$

In order to complete the proof of this lemma, it suffices to show that for $x \ge 8$,

$$G(x) > 0 \tag{3.43}$$

and

$$G(x)-H(x) > 0. \tag{3.44}$$

If (3.43) and (3.44) are verified, we see that the second inequality in (3.35) holds for $x \ge 109$, or equivalently, $n \ge 1204$. The cases for $4 \le n \le 1204$ can be directly verified, and the proof follows.

Thus it remains to verify (3.43) and (3.44). Simple calculations reveal that for $0 \le k \le 96$,

$$-|c_k|x^k > -c_{97}x^{97} \tag{3.45}$$

holds when

$$x > \pi\sqrt{\frac{22+125\pi^2}{2(1+12\pi^2)}} \approx 7.203.$$

Then it follows that for $x \geq 8$,

$$G(x) > -96c_{97}x^{97} + c_{98}x^{98} + c_{99}x^{99}.$$

Since

$$-96c_{97} + c_{98}x + c_{99}x^2 > 0$$

for $x > \sqrt{97+1152\pi^2} + 1 \approx 108.083$, we have $G(x) > 0$ for $x \geq 109$.

Now we turn to prove (3.44). It is easy to check that for $0 \leq k \leq 93$,

$$-|c_k - b_k|x^k > -(c_{94} - b_{94})x^{94}$$

for $x > \frac{\pi}{2}\sqrt{\frac{2432+1824\pi^4+767\pi^6}{2(32+24\pi^4+21\pi^6)}} \approx 7.083$. It immediately follows that

$$G(x) - H(x) > \left(-93(c_{94} - b_{94}) + (c_{95} - b_{95})x + (c_{96} - b_{96})x^2\right)x^{94}.$$

Moreover, we find that for $x > \frac{\sqrt{\frac{3}{2}(992+750\pi^4+651\pi^6)}}{\pi^2} + 3 \approx 106.817$,

$$-93(c_{94} - b_{94}) + (c_{95} - b_{95})x + (c_{96} - b_{96})x^2 > 0.$$

Thus, for $x \geq 107$, $G(x) - H(x) > 0$. □

We are now ready to prove Theorem 3.3.

Proof of Theorem 3.3 It is easy to see that $Q(t)$ is increasing for $0 < t < 1$ since

$$Q'(t) = \frac{1}{(\sqrt{1-t}+1)^3}$$

is positive for $0 < t < 1$. By Theorem 3.1, we know that $f(n) < u_n$ for $n \geq 29$. Then we have for $n \geq 9$,

$$Q(f(n)) < Q(u_n).$$

If we can prove

$$f(n) + \frac{1000}{\mu(n-1)^5} < Q(f(n)) \tag{3.46}$$

for $n \geq 30985$, it is done. Let

$$\psi(t) = Q(t) - t = \frac{3t + 2\sqrt{(1-t)^3} - t^3 - 2}{t^2}.$$

Then (3.46) is equivalent to

$$\psi(f(n)) > \frac{1000}{\mu(n-1)^5}.$$

Since for $0 < t < 1$,

$$\psi'(t) = \frac{\sqrt{1-t}(-t + 3\sqrt{1-t} + 4)}{(\sqrt{1-t}+1)^3} < 0,$$

it is clear that $\psi(t)$ is decreasing for $0 < t < 1$. From (3.35) we see that $0 < f(n) < H(x)/G(x) < 1$ for $n \geq 4$. So it remains to prove

$$\psi\left(f(n)\right) > \psi\left(\frac{H(x)}{G(x)}\right) \quad \text{for} \quad n \geq 30985.$$

Therefore the proof is reduced to prove that for $n \geq 30985$,

$$\psi\left(\frac{H(x)}{G(x)}\right) > \frac{1000}{\mu(n-1)^5}. \tag{3.47}$$

To this end, we should give an estimate for $\psi\left(\frac{H(x)}{G(x)}\right)$. Firstly, we claim that for $x \geq 109$,

$$\frac{\sqrt{5}-1}{2} < \frac{H(x)}{G(x)} < 1. \tag{3.48}$$

To do this, it suffices to show that

$$2H(x) - (\sqrt{5}-1)G(x) \geq 0 \quad \text{for} \quad x \geq 109. \tag{3.49}$$

Notice that

$$b_{97} = c_{97}, \quad b_{98} = c_{98}, \quad b_{99} = c_{99},$$

and observe that for $0 \leq k \leq 96$,

$$-|2b_k - (\sqrt{5}-1)c_k|x^k > -(3-\sqrt{5})c_{97}x^{97}$$

when

$$x > \sqrt{\frac{-125\pi^2\sqrt{5} - 22\sqrt{5} + 381\pi^2 + 66}{2\left(3 - \sqrt{5}\right)\left(1 + 12\pi^2\right)}} \approx 7.42197.$$

Then it follows that for $x \geq 8$,

$$2H(x) - (\sqrt{5} - 1)G(x) > (3 - \sqrt{5})\left(-96c_{97} + c_{98}x + c_{99}x^2\right)x^{97}.$$

Since $-96c_{97} + c_{98}x + c_{99}x^2 > 0$ for $x > \sqrt{97 + 1152\pi^2} + 1 \approx 108.083$, we arrive at (3.49), and so (3.48) holds for $x \geq 109$.

Secondly, we find that

$$\psi(t) < (1 - t)^{3/2} \quad \text{for any } \frac{\sqrt{5} - 1}{2} < t < 1. \tag{3.50}$$

This is because

$$\psi(t) - (1 - t)^{3/2} = \frac{(1 - t)^{3/2}\left(t - \frac{\sqrt{5}-1}{2}\right)\left(t + \frac{\sqrt{5}-1}{2}\right)}{(\sqrt{1 - t} + 1)^2(\sqrt{1 - t} + t)} > 0$$

for $\frac{\sqrt{5}-1}{2} < t < 1$. In view of (3.48) and (3.50), we infer that for $x \geq 109$,

$$\psi\left(\frac{H(x)}{G(x)}\right) > \left(1 - \frac{H(x)}{G(x)}\right)^{3/2}. \tag{3.51}$$

We continue to show that for $x \geq 553$, or equivalently, $n \geq 30985$,

$$\left(1 - \frac{H(x)}{G(x)}\right)^{3/2} > \frac{1000}{\mu(n - 1)^5}. \tag{3.52}$$

Since $G(x) > 0$ for $x \geq 8$, the above inequality can be reformulated as follows. For $x \geq 555$,

$$x^{10}(G(x) - H(x))^3 - 1000^2 G(x)^3 > 0. \tag{3.53}$$

The left-hand side of (3.53) is a polynomial of degree 298, and we write

$$x^{10}(G(x) - H(x))^3 - 1000^2 G(x)^3 = \sum_{k=0}^{298} \gamma_k x^k.$$

The values of γ_{296}, γ_{297} and γ_{298} are given below:

$$\gamma_{296} = 2^{176} \cdot 3^7 \cdot 5^3 \cdot (21\pi^{14} + 96\pi^{12} + 32\pi^8 + 256000000),$$
$$\gamma_{297} = -2^{178} \cdot 3^6 \cdot 5^3 \cdot (32000000 + 9\pi^{12}),$$
$$\gamma_{298} = 2^{177} \cdot 3^6 \cdot 5^3 \cdot \pi^{12}.$$

For $0 \le k \le 295$, we have

$$-|\gamma_k| x^k > -\gamma_{296} x^{296},$$

provided that

$$x > \frac{-2560000000 - 6144000000\pi^2 - 1664\pi^8 - 1776\pi^{12} - 1488\pi^{14} - \pi^{16}}{-1024000000 - 128\pi^8 - 384\pi^{12} - 84\pi^{14}} \approx 36.5822.$$

Thus, for $x \ge 37$,

$$x^{10}(G(x) - H(x))^3 - 1000^2 G(x)^3 > \left(-295\gamma_{296} + \gamma_{297}x + \gamma_{298}x^2\right) x^{296}.$$

The left-hand side of the above inequality is positive, since

$$-295\gamma_{296} + \gamma_{297}x + \gamma_{298}x^2 > 0$$

when

$$x > \frac{\sqrt{\gamma_{297}^2 + 1180\gamma_{296}\gamma_{298}} - \gamma_{297}}{2\gamma_{298}} \approx 552.349.$$

Therefore (3.52) is true. Combining (3.51) and (3.52) yields (3.47) is true for $n \ge 30985$. The proof follows from checking that (3.34) is true for $92 \le n < 30985$ directly. □

With Theorems 3.1, 3.2 and 3.3 in hand, we are ready to give a proof of Theorem 1.2 as follows.

Proof of Theorem 1.2 From (1.6) we know that $u_n < 1$ for $n \ge 2$. Define $F(t)$ to be

$$F(t) = 4(1 - u_n)(1 - t) - (1 - u_n t)^2.$$

Then it is easy to see that the inequality

$$4(1 - u_n)(1 - u_{n+1}) - (1 - u_n u_{n+1})^2 > 0, \quad \text{for} \quad n \ge 16,$$

equivalent to

$$F(u_{n+1}) > 0, \quad \text{for} \quad n \ge 16. \tag{3.54}$$

For $16 \leq n \leq 91$, (3.54) can be easily checked. Therefore, it remains to prove that (3.54) holds for $n \geq 92$. Let $Q(t)$ be as defined in Theorem 3.3, that is

$$Q(t) = \frac{3t + 2\sqrt{(1-t)^3} - 2}{t^2}.$$

Here we first claim that $F(t) > 0$ for $u_n < t < Q(u_n)$. So the proof is reduced to proving that for $n \geq 92$,

$$u_n \leq u_{n+1} \leq Q(u_n).$$

Observe that Wang et al. [18, Theorem 3.1] proved that $u_n < u_{n+1}$ for $n \geq 18$. From Theorem 3.1 we know that $u_{n+1} < g(n+1)$ for $n \geq 92$. Moreover, combining Theorem 3.2 with Theorem 3.3 yields that for $n \geq 92$,

$$g(n+1) < f(n) + \frac{1000}{\mu(n-1)^5} < Q(u_n).$$

Therefore, we conclude that $u_{n+1} < Q(u_n)$ for $n \geq 92$, as required.

Finally, it remains to verify the previous claim. Rewrite $F(t)$ as

$$F(t) = -u_n^2 t^2 + (6u_n - 4)t - 4u_n + 3.$$

The equation $F(t) = 0$ has two solutions

$$P(u_n) = \frac{3u_n - 2\sqrt{(1-u_n)^3} - 2}{u_n^2}, \quad Q(u_n) = \frac{3u_n + 2\sqrt{(1-u_n)^3} - 2}{u_n^2}$$

so that $F(t) > 0$ for $P(u_n) < u_n < Q(u_n)$. Therefore, $F(t) > 0$ for $u_n < t < Q(u_n)$, as claimed. □

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