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A general method for proving the non-trivial linear homogeneous partition inequalities

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Abstract
An asymptotic classification for the linear homogeneous partition inequalities of the form $\sum_{i=1}^{r} p(n+x_i) \leqslant \sum_{i=1}^{s} p(n+y_i)$ has recently been introduced. In this paper, we investigate partition inequalities of this form when $r = s$. From an asymptotic point of view, such partition inequalities are considered to be non-trivial because they have the same number of terms on both sides. In this context, we provide a very general method for proving the non-trivial partition inequalities. This is a numerical method that does not involve q series and connects the non-trivial linear homogeneous partition inequalities with the Prouhet–Tarry–Escott problem: if $\{x_1, x_2, \ldots, x_r\} \stackrel{k}{=} \{y_1, y_2, \ldots, y_r\}$ $(k \geqslant 0)$, then for n large enough the expression $\sum_{i=1}^{r} \big(p(n+x_i) - p(n+y_i)\big)$ has the same sign as $\sum_{i=1}^{r}(x_i^{k+1} - y_i^{k+1})$. The method can be adopted to other inequalities of a similar nature which involve sequences that are asymptotically completely monotone.

Keywords Partition inequalities · Prouhet–Tarry–Escott problem · Finite differences · Algorithms

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1 Introduction

The mathematical theory of partitions began in the mid-eighteenth century when Euler showed that the generating function for $p(n)$, the number of partitions of n, can be expressed as

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$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}, \qquad |q| < 1,$$

where

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

is the q shifted factorial with $(a;q)_0 = 1$. Because the infinite product

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n$$

diverges when $a \neq 0$ and $|q| \geqslant 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q| < 1$.

The Hardy–Ramanujan–Rademacher formula for $p(n)$ states that

$$p(n) = \frac{\sqrt{12}}{24n-1}\sum_{k=1}^{N}\frac{A_k(n)}{\sqrt{k}}\left[\left(1-\frac{k}{\mu(n)}\right)e^{\mu(n)/k} + \left(1+\frac{k}{\mu(n)}\right)e^{-\mu(n)/k}\right] + R(n,N),$$

where $A_k(n)$ is an arithmetic function, $R(n,N)$ is the remainder term and

$$\mu(n) = \frac{\pi}{6}\sqrt{24n-1}$$

(see Hardy and Ramanujan [9], Rademacher [19] for more details). For positive integers n and N, we have the following bound for the remainder term $R(n,N)$ due to Lehmer [13,14]:

$$|R(n,N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}}\left[\left(\frac{N}{\mu(n)}\right)^3 \sin h\left(\frac{\mu(n)}{N}\right) + \frac{1}{6} - \left(\frac{N}{\mu(n)}\right)^2\right]. \qquad (1)$$

Inequalities involving Euler's partition function $p(n)$ have been the subject of recent studies. In [16], the first author proved the inequality

$$p(n) - p(n-1) - p(n-2) + p(n-5) \leqslant 0, \qquad n > 0, \qquad (2)$$

in order to provide the fastest known algorithm for the generation of the partitions of n. Subsequently, Andrews and Merca [1] proved more generally that, for $k > 0$,

$$(-1)^{k-1}\sum_{j=0}^{2k-1}(-1)^{\lceil j/2\rceil}p(n-G_j) \geqslant 0, \qquad (3)$$

where

$$G_k = \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\left(3\left\lceil\frac{k}{2}\right\rceil + (-1)^k\right)$$

is the kth generalized pentagonal number. One can easily verify that (3) reduces to (2) when $k = 2$. In [17], Merca showed that (3) is the case k odd of the inequality

$$(-1)^{\lfloor k/2\rfloor}\sum_{j=0}^{k}(-1)^{\lceil j/2\rceil}p(n-G_j) \geqslant 0.$$

In the same paper, the following two inequalities

$$p(n) - 2p(n-1) + p(n-2) \geqslant 0,$$
$$p(n) - 2p(n-1) + p(n-3) \leqslant 0,$$

are specializations of a more general result: for $n > 1$, $k \geqslant 0$,

$$(-1)^{\lfloor k/2\rfloor}\sum_{j=0}^{k}(-1)^{\lceil j/2\rceil}\big(p(n-G_j) - p(n-1-G_j)\big) \geqslant 0. \tag{4}$$

Very recently, Andrews and Merca [2] proposed the following conjecture: for n odd,

$$(-1)^{k-1}\sum_{j=0}^{2k-1}(-1)^{\lceil j/2\rceil}p(n-T_j) \leqslant (-1)^{k-1}\sum_{j=0}^{2k-1}(-1)^{\lceil j/2\rceil}p(n-G_j), \tag{5}$$

where $T_n = n(n+1)/2$ is the nth triangular number.

Considering Rademacher's convergent series and the Lehmer error bound (1), DeSalvo and Pack [6] investigated in 2015 the log-concavity of Euler's partition function $p(n)$ and confirmed two recent conjectures of Chen [5]:

$$\frac{p(n-1)}{p(n)}\left(1+\frac{1}{n}\right) > \frac{p(n)}{p(n+1)}, \qquad \text{for all} \quad n \geqslant 2 \tag{6}$$

and

$$p(n)^2 - p(n-m)p(n+m) \geqslant 0, \qquad \text{for all} \quad n > m > 1.$$

The inequality (6) has been improved by DeSalvo and Pak [6]:

$$\frac{p(n-1)}{p(n)}\left(1+\frac{240}{(24n)^{3/2}}\right) > \frac{p(n)}{p(n+1)}, \qquad \text{for all} \quad n \geqslant 7.$$

In addition, DeSalvo and Pak [6] proposed the following conjecture:

$$\frac{p(n-1)}{p(n)}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{p(n)}{p(n+1)}, \qquad \text{for all} \quad n \geqslant 45.$$

In 2016, using Lehmer's error bound (1) for the remainder term of $p(n)$, Bessenrodt and Ono [3] proved the following analytic inequality: if a, b are integers with $a, b > 1$ and $a + b > 8$, then

$$p(a)p(b) \geqslant p(a+b).$$

Very recently, Katriel [10] considered the asymptotic behavior of $\frac{p(n+\ell)}{p(n)}$ and provided the following ℓ-dependent partition inequalities:

$$\begin{aligned}
&p(n+\ell) - p(n+\ell-1) \leqslant \big(p(\ell+1) - p(\ell)\big)p(n), && \text{for} \quad n \geqslant 0, \ell \geqslant 1,\\
&p(n+\ell) - p(n+\ell-2) \leqslant \big(p(\ell+1) - p(\ell-1)\big)p(n), && \text{for} \quad n \geqslant 0, \ell \geqslant 1,\\
&p(n+\ell) - p(n+\ell-3) \leqslant \big(p(\ell+1) - p(\ell-2)\big)p(n), && \text{for} \quad n \geqslant 0, \ell \geqslant 2.
\end{aligned}$$

In his paper, Katriel introduced an asymptotic classification for the linear homogeneous partition inequalities of the form

$$\sum_{i=1}^{r} p(n+x_i) \leqslant \sum_{i=1}^{s} p(n+y_i).$$

For $r < s$, it follows from the large n asymptotic behavior of the partition function $p(n)$ that an inequality of this form holds for all $n > N$, for an appropriately specified N. The inequality can be established for all $n \geqslant 1$ by verifying that it holds for the finite set of cases specified by $1 \leqslant n \leqslant N$. Such an inequality has been referred to as *asymptotically trivial*. The asymptotic information provided by the partition function $p(n)$ is not useful when $r = s$. Katriel claimed that in this case a proof has to be provided that applies to all n. Such an inequality has been referred to as *non-trivial*.

In this paper, we investigate the linear homogeneous partition inequalities that Katriel classified as non-trivial, which are of the form

$$\sum_{i=1}^{r} p(n+x_i) \leqslant \sum_{i=1}^{r} p(n+y_i), \tag{7}$$

where $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_r$ are integers. More precisely, we are looking for a criterion to determine the direction of the inequality corresponding to $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_r$. In this way, we provide a numerical method for proving the partition inequalities of the form (7) that does not involve q series and connects the non-trivial linear homogeneous partition inequalities with the Prouhet–Tarry–Escott problem.

Recall that the Prouhet–Tarry–Escott problem asks for two distinct multisets of integers $\{x_1, x_2, \dots, x_r\}$ and $\{y_1, y_2, \dots, y_r\}$ such that

$$\sum_{i=1}^{r} x_i^j = \sum_{i=1}^{r} y_i^j, \quad \text{for all} \quad j = 1, 2, \dots, k,$$

and

$$\sum_{i=1}^{r} x_i^{k+1} \neq \sum_{i=1}^{r} y_i^{k+1},$$

where k is a positive integer. If $k = r - 1$, such a solution is called ideal. In what follows, we consider k a non-negative integer and we write

$$\{x_1, x_2, \dots, x_r\} \stackrel{k}{=} \{y_1, y_2, \dots, y_r\}$$

to denote a solution to the Prouhet–Tarry–Escott problem if k is positive or to denote the case $x_1 + x_2 + \cdots + x_r \neq y_1 + y_2 + \cdots + y_r$ if $k = 0$.

The Prouhet–Tarry–Escott problem has a long history and is, in some form, well over two centuries old. In 1750–1751, Euler and Goldbach noted that

$$\{0, a + b, a + c, b + c\} \stackrel{2}{=} \{a, b, c, a + b + c\}. \tag{8}$$

If a, b, c are distinct positive integers, then it is clear that the coefficients of q^n in

$$\frac{(1 - q^a)(1 - q^b)(1 - q^c)}{(q; q)_\infty}$$

are all non-negative. Thus it is an easy exercise to prove that the partition inequality

$$\begin{aligned} &p(n) + p(n + a + b) + p(n + a + c) + p(n + b + c) \\ &\quad \leqslant p(n + a) + p(n + b) + p(n + c) + p(n + a + b + c) \end{aligned}$$

holds for any non-negative integer n when a, b, c are distinct positive integers. In the general case, we will prove that the direction of the linear homogeneous partition inequality corresponding to (8) is given by the sign of the product abc.

The following results can be used to connect the linear homogeneous partition inequalities of the form (7) with the Prouhet–Tarry–Escott problem.

Lemma 1.1 *Let P be a polynomial of degree N in q which is not divisible by $1 - q$, such that*

$$(1 - q)^k \cdot P(q) = \sum_{n=0}^{N+k} a_{k,n} q^n,$$

where k is a positive integer. For $0 \leqslant p \leqslant k$,

$$\sum_{n=0}^{N+k} (-n)^p a_{k,n} = \delta_{k,p} \cdot k! \cdot P(1),$$

where $\delta_{i,j}$ is the Kronecker delta function.

Corollary 1.1 *Let P be as in Lemma 1.1. Then*

$$\left\{ \underbrace{n, \ldots, n}_{a_{k,n} \text{ times}} | a_{k,n} > 0 \right\} \stackrel{k-1}{=} \left\{ \underbrace{n, \ldots, n}_{|a_{k,n}| \text{ times}} | a_{k,n} < 0 \right\}.$$

Any solution of the Prouhet–Tarry–Escott problem and its partition inequality are related by the following result.

Theorem 1.2 *Let $\{x_1, x_2, \ldots, x_r\} \stackrel{k}{=} \{y_1, y_2, \ldots, y_r\}$. There is a non-negative integer N such that for $n \geqslant N$, the expression*

$$\sum_{i=1}^{r} p(n + x_i) - \sum_{i=1}^{r} p(n + y_i)$$

has the same sign as the expression

$$\sum_{i=1}^{r} x_i^{k+1} - \sum_{i=1}^{r} y_i^{k+1}.$$

We point out that the inequalities (3) and (5) of Andrews and Merca can be derived as special cases of this theorem:

$$\left\{ G_j \,\middle|\, \substack{0 \leqslant j \leqslant 2r-1 \\ j \equiv 0,3 \pmod 4} \right\} \stackrel{0}{=} \left\{ G_j \,\middle|\, \substack{0 \leqslant j \leqslant 2r-1 \\ j \equiv 1,2 \pmod 4} \right\}$$

and

$$\left\{ T_j, G_{j+(-1)^j} \,\middle|\, \substack{0 \leqslant j \leqslant 2r-1 \\ j \equiv 0,3 \pmod 4} \right\} \stackrel{1}{=} \left\{ G_j, T_{j+(-1)^j} \,\middle|\, \substack{0 \leqslant j \leqslant 2r-1 \\ j \equiv 0,3 \pmod 4} \right\}.$$

Having Theorem 1.2, we can say that there is a positive integer $N(k)$, such that the inequality (5) holds for all $n \geqslant N(k)$. On the other hand, for $a \in \{0, 1\}$ we have

$$\left\{ G_j, 1 + G_{j+(-1)^j} \,\middle|\, \substack{0 \leqslant j \leqslant 2r-a \\ j \equiv 0,3 \pmod 4} \right\} \stackrel{a}{=} \left\{ 1 + G_j, G_{j+(-1)^j} \,\middle|\, \substack{0 \leqslant j \leqslant 2r-a \\ j \equiv 0,3 \pmod 4} \right\}.$$

Thus we deduce that the inequality (4) is also a consequence of Theorem 1.2.

It is known that the solutions to the Prouhet–Tarry–Escott problem satisfy many relations. Given a solution to the Prouhet–Tarry–Escott problem, we can generate an infinite family of solutions. That is, if

$$\{x_1, x_2, \dots, x_r\} \stackrel{k}{=} \{y_1, y_2, \dots, y_r\},$$

then

$$\{Mx_1 + N, \dots, Mx_r + N\} \stackrel{k}{=} \{My_1 + N, \dots, My_r + N\}.$$

Thus, without loss of generality, we can consider the non-trivial solution

$$\{x_1, x_2, \dots, x_r\} \stackrel{k}{=} \{y_1, y_2, \dots, y_r\},$$

with

$$0 = x_1 \leqslant x_2 \leqslant \cdots \leqslant x_r \qquad \text{and} \qquad 0 < y_1 \leqslant y_2 \leqslant \cdots \leqslant y_r$$

or

$$0 = x_1 \geqslant x_2 \geqslant \cdots \geqslant x_r \qquad \text{and} \qquad 0 > y_1 \geqslant y_2 \geqslant \cdots \geqslant y_r.$$

We have the following connection between the ideal solutions of the Prouhet–Tarry–Escott problem and the linear homogeneous partition inequalities of the form (7).

Theorem 1.3 *Let* $\{x_1, x_2, \dots, x_r\} \stackrel{r-1}{=} \{y_1, y_2, \dots, y_r\}$ *be an ideal solution of the Prouhet–Tarry–Escott problem with*

$$0 = x_1 \geqslant x_2 \geqslant \cdots \geqslant x_r \qquad \textit{and} \qquad 0 > y_1 \geqslant y_2 \geqslant \cdots \geqslant y_r.$$

There is a non-negative integer N *such that for* $n \geqslant N$*, the coefficients of* q^n *in*

$$\frac{1}{(q;q)_\infty} \sum_{i=1}^{r} (q^{-x_i} - q^{-y_i})$$

are all positive, i.e., for $n \geqslant N$*,*

$$\sum_{i=1}^{r} p(n + x_i) - \sum_{i=1}^{r} p(n + y_i) > 0.$$

On the one hand, it is known that the following relations are equivalent (see [4, Proposition 1]):

(i) $\{x_1, x_2, \dots, x_r\} \stackrel{k-1}{=} \{y_1, y_2, \dots, y_r\}$,
(ii) $(1-q)^k \mid \sum_{i=1}^{r} (q^{-x_i} - q^{-y_i})$.

There is a substantial amount of numerical evidence to conjecture the following result.

Conjecture 1.4 *Let* $\{x_1, x_2, \ldots, x_r\} \stackrel{r-1}{=} \{y_1, y_2, \ldots, y_r\}$ *be an ideal solution of the Prouhet–Tarry–Escott problem with*

$$0 = x_1 \geqslant x_2 \geqslant \cdots \geqslant x_r \quad \text{and} \quad 0 > y_1 \geqslant y_2 \geqslant \cdots \geqslant y_r.$$

The coefficients of q^n *in*

$$\frac{1}{(1-q)^r} \sum_{i=1}^{r} \left(q^{-x_i} - q^{-y_i}\right)$$

are all non-negative.

Some applications of these results will be presented in this paper. In Sects. 3 and 4, we consider the higher-order differences of the partition function $p(n)$. In Sect. 5, we provide a new generalization of the homogeneous linear partition inequality (2). Theorem 1.2 provides a simple algorithm to determine the direction of the partition inequality corresponding to $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_r$ (see Algorithm 1 in the last section of this paper).

2 Proof of Lemma 1.1

We consider $R_k^{(0)}$, a polynomial in q defined as

$$R_k^{(0)}(q) = (1-q)^k P(q) = \sum_{n=0}^{N+k} a_{k,n} q^n.$$

For $p > 0$ we define the polynomial $R_k^{(p)}$ in q as follows:

$$R_k^{(p)}(q) = \begin{cases} \dfrac{\partial}{\partial q} R_k^{(p-1)}(q) & \text{for } p = 1, \\[2ex] \dfrac{\partial}{\partial q}\left(q \cdot R_k^{(p-1)}(q)\right) & \text{for } p > 1. \end{cases}$$

It is clear that

$$R_k^{(p)}(1) = \sum_{n=0}^{N+k} a_{k,n} n^p.$$

On the other hand, the polynomial $R_k^{(p)}$ can be rewritten as

$$R_k^{(p)}(q) = (1-q)^{k-p} Q_k^{(p)}(q), \tag{9}$$

where $Q_k^{(0)}(q) = P_k(q)$. For $p = 1$, we can write

$$R_k^{(1)}(q) = \frac{\partial}{\partial q}\Big((1-q)^k Q_k^{(0)}(q)\Big)$$
$$= -k(1-q)^{k-1} Q_k^{(0)}(q) + (1-q)^k \frac{\partial}{\partial q} Q_k^{(0)}(q). \tag{10}$$

For $1 < p \leqslant k$, we have

$$R_k^{(p)}(q) = \frac{\partial}{\partial q}\Big((1-q)^{k-p+1} q\, Q_k^{(p-1)}(q)\Big)$$
$$= -(k-p+1)(1-q)^{k-p} q\, Q_k^{(p-1)}(q) + (1-q)^{k-p+1} \frac{\partial}{\partial q}\Big(q\, Q_k^{(p-1)}(q)\Big). \tag{11}$$

For $p < k$ we see that $R_k^{(p)}(1) = 0$. For $0 < p \leqslant k$, by (9)–(11) we deduce that $Q_k^{(p)}(1) = -(k-p+1) Q_k^{(p-1)}(1)$. Then having $R_k^{(k)}(1) = Q_k^{(k)}(1)$, we obtain

$$R_k^{(k)}(1) = (-1)^k \cdot k! \cdot Q_k^{(0)}(1)$$

and the proof is finished.

3 Differences of the partition function

The first backward difference of the function f is defined by

$$\nabla f(x) = f(x) - f(x-1).$$

Iterating, we obtain the kth order backward difference:

$$\nabla^k f(x) = \nabla\big(\nabla^{k-1} f(x)\big) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f(x-i).$$

In 1977, Good [7] and Andrews [8] have independently asked about the behavior of $\nabla^k p(n)$. Good conjectured that for each k, there is a positive integer $n_0(k)$ such that $\nabla^k p(n)$ alternates in sign for $n < n_0(k)$, and $\nabla^k p(n)$ is non-negative for $n \geqslant n_0(k)$. In 1978, Gupta [8] considered the Hardy–Ramanujan–Rademacher formula for $p(n)$ and proved that $\nabla^k p(n) > 0$ for each fixed k if n is sufficiently large. In 1988, Odlyzko [18] proved the conjecture of Good and obtained the following asymptotic formula for $n_0(k)$:

$$n_0(k) \sim \frac{6}{\pi^2} k^2 \log k^2 \quad \text{as} \quad k \to \infty.$$

This asymptotic result is not very accurate for moderate values of k because the convergence to the limit is very slow. In 1990, Knessl and Keller [11,12] obtained a better asymptotic approximation $n_0'(k)$ of $n_0(k)$ by deriving a recurrence for $\nabla^k p(n)$ and solving it asymptotically. For example,

$$\left|n_0'(k) - n_0(k)\right| \leqslant 2 \qquad \text{for all} \qquad k \leqslant 75.$$

For $n \geqslant n_0(k)$, the inequality

$$\nabla^k p(n) \geqslant 0$$

can be rewritten as a linear homogeneous partition inequality of the form (7):

$$\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i+1} p(n-1-2i) \leqslant \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i} p(n-2i). \tag{12}$$

Moreover, considering the case $P = 1$ of Lemma 1.1, we deduce the following solution for the Prouhet–Tarry–Escott problem

$$\left\{ \underbrace{-2i-1, \ldots, -2i-1}_{\binom{k}{2i+1} \text{ times}} | i = 0, 1, \ldots, \lfloor k/2 \rfloor \right\}$$
$$\stackrel{k-1}{=} \left\{ \underbrace{-2i, \ldots, -2i}_{\binom{k}{2i} \text{ times}} | i = 0, 1, \ldots, \lfloor k/2 \rfloor \right\}.$$

and the following identity

$$\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i+1} (-2i-1)^k - \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i} (-2i)^k = -k!. \tag{13}$$

We remark that the direction of the inequality (12) is given by the identity (13), i.e., for all $n \geqslant n_0(k)$ the expression

$$\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i+1} p(n-1-2i) - \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i} p(n-2i)$$

has the same sign as

$$\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i+1} (-2i-1)^k - \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i} (-2i)^k.$$

Example 1 Considering Algorithm 1, we illustrate the connection between (12) and (13) when $k \in \{2, 3, 4\}$.

1. For $k = 2$, we have the ideal solution

$$\{-1, -1\} \overset{1}{=} \{0, -2\}.$$

The partition inequality

$$2p(n-1) \leqslant p(n) + p(n-2).$$

holds for all $n \geqslant 2$. According to [12, Table 1], the initial value of N in Algorithm 1 is $n_0(2) = 2$.

2. For $k = 3$, we have the solution

$$\{-1, -1, -1, -3\} \overset{2}{=} \{0, -2, -2, -2\}.$$

The partition inequality

$$3p(n-1) + p(n-3) \leqslant p(n) + 3p(n-2)$$

holds for all $n \geqslant 26$. According to [12, Table 1], the initial value of N in Algorithm 1 is $n_0(3) = 26$.

3. For $k = 4$, we have the solution

$$\{-1, -1, -1, -1, -3, -3, -3, -3\} \overset{3}{=} \{0, -2, -2, -2, -2, -2, -2, -4\}.$$

The partition inequality:

$$4p(n-1) + 4p(n-3) \leqslant p(n) + 6p(n-2) + p(n-4)$$

holds for all $n \geqslant 68$. According to [12, Table 1], the initial value of N in Algorithm 1 is $n_0(4) = 68$.

4 Partitions into parts greater than *m*

The results presented in the previous section can be extended to the partitions of n into parts greater than m. We denote by $p_m(n)$ the number of these partitions. Noting that

$$\sum_{n=0}^{\infty} p_m(n)q^n = \frac{(q;q)_m}{(q;q)_\infty},$$

we easily obtain the following expression for the generating function of $\nabla^k p_m(n)$:

$$\sum_{n=0}^{\infty} \nabla^k p_m(n) q^n = \frac{(1-q)^k (q;q)_m}{(q;q)_\infty}.$$

On the one hand, we know that for each fixed k there is a positive integer $n_0(k)$ such that the coefficients of q^n in $\dfrac{(1-q)^k}{(q;q)_\infty}$ are positive for $n \geqslant n_0(k)$. On the other hand, we have

$$\frac{(1-q)^k (q;q)_m}{(q;q)_\infty} = \frac{(1-q)^{k+m}}{(q;q)_\infty} \cdot P(q),$$

where

$$P(q) = \prod_{j=1}^{m-1} (1 + q + q^2 + \cdots + q^j)$$

is a polynomial in q with positive coefficients. In this way, we deduce that for each fixed k and m there is a positive integer $n_0(k, m)$ such that

$$\nabla^k p_m(n) \geqslant 0 \qquad \text{for all} \qquad n \geqslant n_0(k, m).$$

Having

$$\sum_{j=0}^{k+\binom{m+1}{2}} a_{k,m,j} q^j = (1-q)^{k+m} \cdot P(q),$$

it is clear that the linear homogeneous partition inequality

$$\sum_{j=0}^{k+\binom{m+1}{2}} a_{k,m,j}\, p(n-j) > 0,$$

holds for $n \geqslant n_0(k, m)$. This inequality is of the form (7). In addition, by Lemma 1.1 we obtain

$$\sum_{j=0}^{k+\binom{m+1}{2}} (-j)^p a_{k,m,j} = \delta_{k+m,p} \cdot (k+m)! \cdot m!, \qquad p \leqslant k+m. \tag{14}$$

In the spirit of Theorem 1.2, we notice that for all $n \geqslant n_0(k,m)$ the expression

$$\sum_{j=0}^{k+\binom{m+1}{2}} a_{k,m,j}\, p(n-j)$$

is positive just as the expression

$$\sum_{j=0}^{k+\binom{m+1}{2}} (-j)^{k+m} a_{k,m,j}.$$

Example 2 Considering Algorithm 1, we point out a few special cases of (14) as ideal solutions for the Prouhet–Tarry–Escott problem and their partition inequalities:

1. For $k=0$ and $m=2$, we have

$$\{a_{0,2,n}\}_{0 \leqslant n \leqslant 3} = \{1, -1, -1, 1\}.$$

   So we obtain the ideal solution

$$\{0, -3\} \stackrel{1}{=} \{-1, -2\}$$

   with the partition inequality

$$p(n) + p(n-3) \geqslant p(n-1) + p(n-2)$$

   that holds for all $n \geqslant 0$. According to [12, Table 1], the initial value of N in Algorithm 1 is $n_0(3) = 26$.
2. For $k=0$ and $m=3$, we have

$$\{a_{0,3,n}\}_{0 \leqslant n \leqslant 6} = \{1, -1, -1, 0, 1, 1, -1\}.$$

   We obtain the ideal solution

$$\{0, -4, -5\} \stackrel{2}{=} \{-1, -2, -6\}$$

   with the partition inequality

$$p(n) + p(n-4) + p(n-5) \geqslant p(n-1) + p(n-2) + p(n-6)$$

   that holds for all $n \geqslant 0$. According to [12, Table 1], the initial value of N in Algorithm 1 is $n_0(6) = 228$.
3. For $k=0$ and $m=4$, we have

$$\{a_{0,4,n}\}_{0 \leqslant n \leqslant 10} = \{1, -1, -1, 0, 0, 2, 0, 0, -1, -1, 1\}.$$

We obtain the ideal solution

$$\{0,-5,-5,-10\} \stackrel{3}{=} \{-1,-2,-8,-9\}$$

with the inequality

$$p(n)+2p(n-5)+p(n-10) \geqslant p(n-1)+p(n-2)+p(n-8)+p(n-9)$$

that holds for all $n \geqslant 0$. According to [12, Table 1], the initial value of N in Algorithm 1 is $n_0(10)=934$.

4. For $k=1$ and $m=2$, we have

$$\{a_{1,2,n}\}_{0\leqslant n\leqslant 4} = \{1,-2,0,2,-1\}.$$

We obtain the ideal solution

$$\{0,-3,-3\} \stackrel{2}{=} \{-1,-1,-4\}$$

with the partition inequality

$$p(n)+2p(n-3) \geqslant 2p(n-1)+p(n-4)$$

that holds for all $n \geqslant 2$. According to [12, Table 1], the initial value of N in Algorithm 1 is $n_0(4)=68$.

5. For $k=1$ and $m=3$, we have

$$\{a_{1,3,n}\}_{0\leqslant n\leqslant 7} = \{1,-2,0,1,1,0,-2,1\}.$$

We obtain the ideal solution

$$\{0,-3,-4,-7\} \stackrel{3}{=} \{-1,-1,-6,-6\}$$

with the partition inequality

$$p(n)+p(n-3)+p(n-4)+p(n-7) \geqslant 2p(n-1)+2p(n-6)$$

that holds for all $n \geqslant 2$. According to [12, Table 1], the initial value of N in Algorithm 1 is $n_0(7)=352$.

6. For $k=1$ and $m=5$, we have

$$\{a_{1,5,n}\}_{0\leqslant n\leqslant 16} = \{1,-2,0,1,0,1,0,0,-2,0,0,1,0,1,0,-2,1\}.$$

We obtain the ideal solution

$$\{0,-3,-5,-11,-13,-16\} \stackrel{5}{=} \{-1,-1,-8,-8,-15,-15\}$$

with the partition inequality

$$
\begin{aligned}
&p(n)+p(n-3)+p(n-5)+p(n-11)+p(n-13)+p(n-16)\\
&\quad\geqslant 2p(n-1)+2p(n-8)+2p(n-15)
\end{aligned}
$$

that holds for all $n \geqslant 2$. According to [12, Table 1], the initial value of N in Algorithm 1 is $n_0(16) = 3188$.

5 A generalization of the inequality (2)

In this section, we consider the following special case of Lemma 1.1:

$$
P(q) = (1-q-q^{k+2})\prod_{j=1}^{k}(1+q+q^2+\cdots+q^j),
$$

with $k \geqslant 1$. Thus we have

$$
\sum_{n=0}^{\binom{k+3}{2}-1} a_{k,n}q^n = (1-q-q^{k+2})(q^2;q)_k.
$$

Considering Euler's formula

$$
\sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty},
$$

we can write

$$
\begin{aligned}
\frac{(1-q-q^{k+2})(q^2;q)_k}{(q;q)_\infty} &= \frac{1-q-q^{k+2}}{(1-q)(q^{k+2};q)_\infty}\\
&= \frac{1}{(q^{k+2};q)_\infty} - \frac{q^{k+2}}{(1-q)(q^{k+2};q)_\infty}\\
&= \sum_{j=0}^{\infty}\frac{q^{(k+2)j}}{(q;q)_j} - \sum_{j=0}^{\infty}\frac{q^{(k+2)(j+1)}}{(1-q)(q;q)_j}\\
&= 1+\sum_{j=1}^{\infty}\frac{q^{(k+2)j}}{(q;q)_j} - \sum_{j=1}^{\infty}\frac{q^{(k+2)j}(1-q^j)}{(1-q)(q;q)_j}\\
&= 1+\sum_{j=2}^{\infty}\frac{q^{(k+2)j}}{(q;q)_j}\left(1-\frac{1-q^j}{1-q}\right)\\
&= 1-\sum_{j=2}^{\infty}\frac{q^{(k+2)j}}{(q;q)_j}\left(q+q^2+\cdots+q^{j-1}\right).
\end{aligned}
$$

We see that the coefficient of q^0 equals 1, the coefficients of q^j for $1 \leqslant j \leqslant 2(k+2)$ are all 0, and for $j > 2(k+2)$ all the coefficients are negative. In this way, we proved that the inequality

$$\sum_{j=0}^{\binom{k+3}{2}-1} a_{k,j}\, p(n-j) \leqslant 0 \tag{15}$$

holds for all $n \geqslant 1$. For $n > 2(k+2)$ we have a strict inequality. One can easily verify that the inequality (15) reduces to (2) when $k = 1$. In addition, taking into account that

$$\frac{(1-q-q^{k+2})(q^2;q)_k - (1-q-q^{k+3})(q^2;q)_{k+1}}{(q;q)_\infty} = \frac{-q^{2k+5}}{(1-q)(q^{k+2};q)_\infty},$$

we deduce that the inequality

$$\sum_{j=0}^{\binom{k+4}{2}-1} (a_{k+1,j} - a_{k,j})\, p(n-j) \geqslant 0 \tag{16}$$

holds for all $n \geqslant 0$, with strict inequality if $n \geqslant 2k+5$. For $j \geqslant \binom{k+3}{2}$, we set $a_{k,j} = 0$.

The inequalities (15) and (16) are of the form (7). By Lemma 1.1, we obtain the following identity:

$$\sum_{j=0}^{\binom{k+3}{2}-1} (-j)^p a_{k,j} = -\delta_{k,p} \cdot (k+1)! \cdot k!, \qquad p \leqslant k. \tag{17}$$

Then we derive

$$\sum_{j=0}^{\binom{k+4}{2}-1} (-j)^p (a_{k+1,j} - a_{k,j}) = \delta_{k,p} \cdot (k+1)! \cdot k!, \qquad p \leqslant k. \tag{18}$$

It is clear that for $n \geqslant 2(k+2)$ the expression

$$\sum_{j=0}^{\binom{k+3}{2}-1} a_{k,j}\, p(n-j)$$

is negative just as the expression

$$\sum_{j=0}^{\binom{k+3}{2}-1} (-j)^k a_{k,j}.$$

Similarly, for $n \geqslant 2k+5$ the expression

$$\sum_{j=0}^{\binom{k+4}{2}-1} (a_{k+1,j} - a_{k,j}) p(n-j)$$

is positive just as the expression

$$\sum_{j=0}^{\binom{k+4}{2}-1} (-j)^k (a_{k+1,j} - a_{k,j}).$$

Example 3 We point out the case $k = 2$ of (17) as a solution for the Prouhet–Tarry–Escott problem and its partition inequality. We have

$$\{a_{2,n}\}_{0 \leqslant n \leqslant 9} = \{1, -1, -1, 0, 0, 1, 0, 1, 0, -1\}.$$

So we obtain the solution

$$\{0, -5, -7\} \stackrel{1}{=} \{-1, -2, -9\}$$

with the partition inequality

$$p(n) + p(n-5) + p(n-7) \leqslant p(n-1) + p(n-2) + p(n-9)$$

that holds for all $n \geqslant 1$, with strict inequality if $n \geqslant 9$.

Example 4 Considering (18) with $k = 2$, we have

$$\{a_{3,n} - a_{2,n}\}_{0 \leqslant n \leqslant 14} = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -1, -1, 0, 1\}.$$

So we obtain the solution

$$\{-9, -14\} \stackrel{1}{=} \{-11, -12\}$$

with the partition inequality

$$p(n-9) + p(n-14) > p(n-11) + p(n-12)$$

that holds for all $n \geqslant 9$ or

$$p(n) + p(n-5) > p(n-2) + p(n-3)$$

that holds for all $n \geqslant 0$.

6 Proof of Theorem 1.2

The first forward difference of the function f is defined by

$$\Delta f(x) = f(x+1) - f(x).$$

Iterating, we obtain the kth order forward difference:

$$\Delta^k f(x) = \Delta\big(\Delta^{k-1} f(x)\big) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x+i).$$

Considering Newton's forward difference formula, Euler's partition function $p(n)$ can be expressed in terms of the kth order forward difference as follows:

$$p(n+\xi) = \sum_{j=0}^{\infty} \frac{\Delta^j p(n)}{j!} (\xi)_j,$$

where

$$(\xi)_j = \xi(\xi-1)\cdots(\xi-j+1)$$

is the falling factorial with $(\xi)_0 = 1$. Then we can write:

$$\sum_{i=1}^{r} \big(p(n+x_i) - p(n+y_i)\big) = \sum_{j=0}^{\infty} \frac{\Delta^j p(n)}{j!} \sum_{i=1}^{r} \big((x_i)_j - (y_i)_j\big).$$

Without loss of generality, we consider that $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_r$ are non-negative. By $\{x_1, x_2, \ldots, x_r\} \stackrel{k}{=} \{y_1, y_2, \ldots, y_r\}$, we deduce that

(i) $\displaystyle\sum_{i=1}^{r} (x_i)_j = \sum_{i=1}^{r} (y_i)_j \quad$ for all $\quad j = 0, 1, \ldots, k$;

(ii) $\displaystyle\sum_{i=1}^{r} (x_i)_j \neq \sum_{i=1}^{r} (y_i)_j \quad$ for $\quad j > k$;

(iii) $\displaystyle\sum_{i=1}^{r} \big((x_i)_{k+1} - (y_i)_{k+1}\big) = \sum_{i=1}^{r} (x_i^{k+1} - y_i^{k+1})$.

Let $M = \max(x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_r)$. For $j > M$, it is clear that

$$\sum_{i=1}^{r} \big((x_i)_j - (y_i)_j\big) = 0.$$

Thus we obtain

$$\sum_{i=1}^{r}\big(p(n+x_i)-p(n+y_i)\big)=\sum_{j=k+1}^{M}\frac{\Delta^j p(n)}{j!}\sum_{i=1}^{r}\big((x_i)_j-(y_i)_j\big).$$

In addition, for $k<j\leqslant M$ the expression

$$\sum_{i=1}^{r}\big((x_i)_j-(y_i)_j\big)$$

has the same sign as (iii), because for $j>k$ the expression

$$\sum_{i=1}^{r}\left(x_i^j-y_i^j\right)$$

has the same sign as (iii). Moreover, it is known that for each fixed j there is a positive integer $n_0(j)$ such that $\Delta^j p(n)\geqslant 0$. Clearly for

$$n\geqslant \max\big(n_0(k+1),n_0(k+2),\ldots,n_0(M)\big),$$

the expression

$$\sum_{i=1}^{r}\big(p(n+x_i)-p(n+y_i)\big)$$

has the same sign as (iii). The proof of Theorem 1.2 is finished.

7 Proof of Theorem 1.3

Considering the following lemma, Theorem 1.3 can be derived as a specialization of Theorem 1.2.

Lemma 7.1 *Let* $\{x_1,x_2,\ldots,x_r\}\overset{r-1}{=}\{y_1,y_2,\ldots,y_r\}$ *be an ideal solution for the Prouhet–Tarry–Escott problem with*

$$0=x_1\leqslant x_2\leqslant\ldots\leqslant x_r\quad\text{and}\quad 0<y_1\leqslant y_2\leqslant\ldots\leqslant y_r.$$

Then

$$\sum_{i=1}^{r}x_i^r-\sum_{i=1}^{r}y_i^r=(-1)^r r\prod_{i=1}^{r}y_i.$$

Proof To prove this result, we use several tools from symmetric functions theory. We work with formal symmetric functions and we will use the standard notation from Macdonald's book [15] for the classical families of symmetric functions: e_k for the kth elementary symmetric function

$$e_k(\xi_1, \xi_2, \ldots, \xi_n) = \sum_{1 \leqslant n_1 < n_2 < \ldots < n_k \leqslant n} \xi_{n_1} \xi_{n_2} \ldots \xi_{n_k}$$

and p_k for the k-th power sum symmetric function

$$p_k(\xi_1, \xi_2, \ldots, \xi_n) = \sum_{j=1}^{n} \xi_j^k.$$

The following relation is well known as Newton's identity:

$$k e_k(\xi_1, \xi_2, \ldots, \xi_n) = \sum_{j=1}^{k} (-1)^{j-1} e_{k-j}(\xi_1, \xi_2, \ldots, \xi_n) p_j(\xi_1, \xi_2, \ldots, \xi_n). \tag{19}$$

Having

$$p_j(x_1, x_2, \ldots, x_r) = p_j(y_1, y_2, \ldots, y_r) \qquad \text{for all } j = 1, 2, \ldots, r-1,$$

by (19) we deduce that

$$e_j(x_1, x_2, \ldots, x_r) = e_j(y_1, y_2, \ldots, y_r) \qquad \text{for all } j = 1, 2, \ldots, r-1.$$

Using again (19), we obtain

$$\begin{aligned} & r\big(e_r(x_1, x_2, \ldots, x_r) - e_r(y_1, y_2, \ldots, y_r)\big) \\ & \quad = (-1)^{r-1}\big(p_r(x_1, x_2, \ldots, x_r) - p_r(y_1, y_2, \ldots, y_r)\big). \end{aligned}$$

Taking into account that $e_r(x_1, x_2, \ldots, x_r) = 0$, the proof is finished. □

8 Concluding remarks

Interesting connections between the Prouhet–Tarry–Escott problem and the linear homogeneous partition inequalities of the form (7) have been introduced in this paper. A criterion to determine the direction of the linear homogeneous partition inequality corresponding to an ideal solution of the Prouhet–Tarry–Escott problem has been provided by Theorem 1.3. A more general criterion referring to the direction of any linear homogeneous partition inequality of the form (7) was provided by Theorem 1.2. Surprisingly, connections between linear partition inequalities and the Prouhet–Tarry–Escott problem have not been noticed so far.

Any non-trivial linear homogeneous partition inequality is equivalent to an inequality of the form (7) with

$$0 \geqslant x_1 \geqslant x_2 \geqslant \cdots \geqslant x_r \quad \text{and} \quad 0 > y_1 \geqslant y_2 \geqslant \cdots \geqslant y_r.$$

According to the proof of Theorem 1.2, we provide the following algorithm to determine the direction of the partition inequality corresponding to $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_r$.

Algorithm 1 Proving non-trivial partition inequalities

Require: $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_r$ and $y_1 \geqslant y_2 \geqslant \ldots \geqslant y_r$ (non-positive integers)
Ensure: N, S

1: $k \leftarrow 0$
2: $S \leftarrow 0$
3: **while** $S = 0$ **do**
4: $\quad k \leftarrow k + 1$
5: $\quad S \leftarrow \sum_{i=1}^{r} (x_i^k - y_i^k)$
6: **end while**
7: $M \leftarrow \max(|x_r|, |y_r|)$
8: $N \leftarrow n_o(M)$
9: **while** $\left(S \cdot \sum_{i=1}^{r} \big(p(N-1+x_i) - p(N-1+y_i)\big) \geqslant 0 \right)$ **and** $(N > 0)$ **do**
10: $\quad N \leftarrow N - 1$
11: **end while**

In the line (8) of Algorithm 1, the parameter N is initialized in accordance with Knessl and Keller [12]. If Algorithm 1 returns a positive value for S then the inequality

$$\sum_{i=1}^{r} p(n + x_i) \geqslant \sum_{i=1}^{r} p(n + y_i)$$

holds for any $n \geqslant N$. When Algorithm 1 returns a negative value for S then the inequality

$$\sum_{i=1}^{r} p(n + x_i) \leqslant \sum_{i=1}^{r} p(n + y_i)$$

holds for any $n \geqslant N$.

Finally, we remark that Theorem 1.2 and Algorithm 1 can be adopted to other partition functions and other infinite sequences that are asymptotically completely monotone.

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