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Inequalities for the overpartition function arising from determinants

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ABSTRACT

Let $\overline{p}(n)$ denote the overpartition function. This paper presents the 2-log-concavity property of $\overline{p}(n)$ by considering a more general inequality of the following form

$$\begin{vmatrix} \overline{p}(n) & \overline{p}(n+1) & \overline{p}(n+2) \\ \overline{p}(n-1) & \overline{p}(n) & \overline{p}(n+1) \\ \overline{p}(n-2) & \overline{p}(n-1) & \overline{p}(n) \end{vmatrix} > 0,$$

which holds for all $n \geq 42$.

1. Introduction

A sequence $(a_n)_{n\geq 0}$ is called log-concave if it satisfies $a_n^2 \geq a_{n-1}a_{n+1}$ for all $n \geq 1$. The binomial coefficients, the Eulerian numbers, the Stirling numbers are well known combinatorial sequences which are log-concave. For a more detailed exposition on log-concavity of sequences, we refer the reader to [2,22]. The study of log-concavity property of sequences shares an intimate connection with zeros of polynomials. In this regard, Newton proved the following theorem.

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Theorem 1.1. *[11, p. 52] Let*

$$P(x) = \sum_{j=0}^{n} \binom{n}{j} a_j x^j$$

be a (real) polynomial with real zeros. Then for all $1 \leq j \leq n-1$

$$a_j^2 \geq a_{j-1}a_{j+1}\Big(\frac{j+1}{j}\Big)\Big(\frac{n-j+1}{n-j}\Big),$$

and in particular, $(a_k)_{0\leq k\leq n}$ *is log-concave.*

Now we shall look at the log-concavity property of a sequence $(a_n)_{n\geq 0}$ from two different directions, namely, real rootedness of Jensen polynomial associated with the sequence and total positivity of the sequence. The Jensen polynomial of degree d associated with a sequence $(a_n)_{n\geq 0}$ is defined by

$$J_a^{d,n}(x) := \sum_{j=0}^{d} \binom{d}{j} a(n+j)x^j.$$

It is easy to observe that $J_a^{2,n}(x)$ (resp. $J_a^{3,n}(x)$) has all real roots if and only if $(a_n)_{n\geq 0}$ is log-concave (resp. satisfies the higher order Turán inequalities[1]). Before we introduce the theory of total positivity of a sequence, let us describe the theory of total positivity for matrices. A matrix A with entries in real number is called totally positive of order r if the determinant of each of its minors of order t is nonnegative for all $1 \leq t \leq r$ and the matrix is called totally positive if it is totally positive for all $r \geq 1$. For a sequence $(a_n)_{n\geq 0}$, define the associated Toeplitz matrix $T = (T_{i,j})_{i,j\geq 0} := (a_{i-j})_{i,j\geq 0}$ with $a_\ell = 0$ for $\ell < 0$, that is,

$$T = \begin{pmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Consequently, $(a_n)_{n\geq 0}$ is called a totally positive sequence of order r (or Pólya frequency sequence of order r) if its associated Toeplitz matrix is totally positive of order r. Note that, a totally positive sequence of order 2 is log-concave. Moreover, $(a_n)_{n\geq 0}$ is called totally positive if it is totally positive of order r for any $r \geq 1$. We end this short discussing by stating a more general version of Newton's result (cf. Theorem 1.1) due to Aissen et al. [1].

[1] $(a_n)_{n\geq 0}$ satisfies the higher order Turán inequalities if $\mathrm{disc}_x(J_a^{3,n}(x)) \geq 0$.

Theorem 1.2. *Let $P(x) = \sum_{i=0}^{d} a_j x^j$ be a real polynomial with nonnegative coefficients. Then $P(x)$ has only real zeros if and only if its coefficient sequence is a Pólya frequency sequence.*

Consider the operator $\mathcal{L}$ defined on a sequence $a := (a_n)_{n\geq 0}$ by $\mathcal{L}(a) := (a_n^2 - a_{n-1}a_{n+1})_{n\geq 1}$. Hence the sequence $(a_n)_{n\geq 0}$ is log-concave if and only if $\mathcal{L}(a)$ is a nonnegative sequence. The sequence $(a_n)_{n\geq 0}$ is called 2-log-concave if $\mathcal{L}^j(a)$ is a nonnegative sequence for all $0 \leq j \leq 2$. The binomial coefficients are 2-log-concave. Note that, for the Jensen polynomial of degree 4 associated with a sequence $(a_n)_{n\geq 0}$, the corresponding quartic binary form

$$Q_4(x,y) = \sum_{j=0}^{4} \binom{4}{j} a(n+j) x^j y^{4-j}$$

has two invariants and one of them is

$$I_a(n) := -a_n a_{n+2} a_{n+4} + a_{n+2}^3 + a_n a_{n+3}^2 + a_{n+1}^2 a_{n+4} - 2a_{n+1}a_{n+2}a_{n+3}.$$

Nonnegativity of $I_a(n)$ implies that $(a_n)_{n\geq 0}$ is 2-log-concave. From the framework of total positivity, it is immediate that $(a_n)_{n\geq 0}$ is 2-log-concave if and only if it is totally positive of order 3. The notion of 2-log-concavity of a sequence can be derived from Pólya and Schur's work [20] on multiplier sequence. For a more detailed study in this direction, we refer the reader to [7].

A partition of a positive integer n is a nonincreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ of positive integers with $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$ and $p(n)$ denotes the number of partitions of n. The systematic study of partitions dates back to Euler. Rigorous analytic approach comes into play in the theory of partitions since the foundational work of Hardy and Ramanujan [12]. Hardy and Ramanujan employed the celebrated circle method in order to explicitly describe the asymptotics of $p(n)$, specifically, given by

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}} \quad (n \to \infty). \tag{1.1}$$

Later Rademacher [21] refined the formulation of Hardy and Ramanujan to set a convergent series expression for $p(n)$ and an error bound was given due to Lehmer [16]. Log-concavity of $p(n)$ has been studied independently by Nicolas [19] and by DeSalvo and Pak [8] by confirming a conjecture of Chen [3]. Since then the study on inequalities of the partition function from combinatorial analysis perspective has been documented in the works of Chen et al. [4], [5]. More generally, Griffin, Ono, Rolen, and Zagier [10] proved that $J_p^{d,n}(x)$ has all real roots for sufficiently large n. Later Larson and Wagner [15] provided an estimate for the cut off $N(d)$ such that for all $n \geq N(d)$, $J_p^{d,n}(x)$ has all real roots. Jia and Wang [14] studied determinantal inequalities for $p(n)$ arising from the theory of total positivity.

Theorem 1.3 *(Theorem 1.5, [14]). Let $p(n)$ denote the partition function and*

$$M_3(p(n)) = \begin{pmatrix} p(n) & p(n+1) & p(n+2) \\ p(n-1) & p(n) & p(n+1) \\ p(n-2) & p(n-1) & p(n) \end{pmatrix} \tag{1.2}$$

Then for all $n \geq 222$, we have

$$\det M_3(p(n)) > 0. \tag{1.3}$$

As a corollary of Theorem 1.3, they proved 2 log-concavity of $p(n)$. Independently, Hou and Zhang [13] proved that $(p(n))_{n\geq 222}$ is 2-log-concave.

Corteel and Lovejoy [6] initiated a broad generalization of partitions, called overpartition that offers a panorama of combinatorial perspective of basic hypergeometric series. An overpartition of n is a nonincreasing sequence of natural numbers whose sum is n in which the first occurrence (equivalently, the final occurrence) of a number may be overlined and $\overline{p}(n)$ denotes the number of overpartitions of n. For convenience, define $\overline{p}(0) = 1$. For example, there are 8 overpartitions of 3 enumerated by $3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1$. Similar to the Hardy-Ramanujan-Rademacher type formula for $\overline{p}(n)$, Zuckerman [25] showed that

$$\overline{p}(n) = \frac{1}{2\pi} \sum_{\substack{k=1 \\ 2\nmid k}}^{\infty} \sqrt{k} \sum_{\substack{h=0 \\ (h,k)=1}}^{k-1} \frac{\omega(h,k)^2}{\omega(2h,k)} e^{-\frac{2\pi i n h}{k}} \frac{d}{dn}\left(\frac{\sinh \frac{\pi\sqrt{n}}{k}}{\sqrt{n}}\right), \tag{1.4}$$

where

$$\omega(h,k) = \exp\left(\pi i \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right)\right)$$

for positive integers $(h,k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$. In order to prove log-concavity of $\overline{p}(n)$, Engel [9] provided an error term for $\overline{p}(n)$ as follows:

$$\overline{p}(n) = \frac{1}{2\pi} \sum_{\substack{k=1 \\ 2\nmid k}}^{N} \sqrt{k} \sum_{\substack{h=0 \\ (h,k)=1}}^{k-1} \frac{\omega(h,k)^2}{\omega(2h,k)} e^{-\frac{2\pi i n h}{k}} \frac{d}{dn}\left(\frac{\sinh \frac{\pi\sqrt{n}}{k}}{\sqrt{n}}\right) + R_2(n,N), \tag{1.5}$$

where

$$\left|R_2(n,N)\right| < \frac{N^{5/2}}{\pi n^{3/2}} \sinh\left(\frac{\pi\sqrt{n}}{N}\right). \tag{1.6}$$

Along the lines of works of Chen et al. in context of the partition function, somewhat similar research works on inequalities for $\overline{p}(n)$ has already been recorded in [23] and [17].

In this paper, our primary goal is to prove 2-log-concavity of $\overline{p}(n)$. In order to prove this, we set up a similar device as that of Theorem 1.3 but in context of overpartitions. In particular, we shall prove the following result.

Theorem 1.4. *Let $\overline{p}(n)$ denote the overpartition function. Then for all $n \geq 42$, we have*

$$\begin{vmatrix} \overline{p}(n) & \overline{p}(n+1) & \overline{p}(n+2) \\ \overline{p}(n-1) & \overline{p}(n) & \overline{p}(n+1) \\ \overline{p}(n-2) & \overline{p}(n-1) & \overline{p}(n) \end{vmatrix} > 0. \tag{1.7}$$

Remark 1.5. Two independent proofs of Theorem 1.4 can be found in [24, Theorem 5.3] and [18, Corollary 1.7].

Theorem 1.4 straight away implies the 2-log-concavity of $\overline{p}(n)$, precisely

Theorem 1.6. *For all $n \geq 42$,*

$$\left(\overline{p}(n)^2-\overline{p}(n-1)\overline{p}(n+1)\right)^2-\left(\overline{p}(n-1)^2-\overline{p}(n-2)\overline{p}(n)\right)\left(\overline{p}(n+1)^2-\overline{p}(n)\overline{p}(n+2)\right) > 0. \tag{1.8}$$

We organize this paper in the following format. In Section 2, we set up the premise first by introducing an inequality for $\overline{p}(n)$ and reformulate Theorem 1.4, given in Theorem 2.5, followed by the documentation of Theorem 2.6 and 2.7. This foundation enables us to provide the proof of Theorem 2.5 in Section 3. In the end, we discuss how one can guess an infinite number of inequalities for the overpartition function by considering totally positive matrix of order k with $k \in \mathbb{Z}_{\geq 2}$, described in Problem 4.1.

2. Inequality for $\overline{p}(n)$ and its consequences

The principal aim of this section is to construct the machinery in order to prove Theorem 1.4, the primary objective of this paper. To some extent, we follow a similar line of argument as described in the work of Jia and Wang [14]. We will see that the Theorem 2.6 and 2.7 are the key tools to prove Theorem 2.5, a reprise version of Theorem 1.4. First we need to estimate the quotient $u_n = \frac{\overline{p}(n-1)\overline{p}(n+1)}{\overline{p}(n)^2}$ by showing its upper and lower bound $g(n)$ and $f(n)$ respectively (cf. Theorem 2.3), derived from the inequality $B_1(n) < \overline{p}(n) < B_2(n)$ with $B_1(n), B_2(n)$ given in (2.1) and as an immediate consequence, we get the inequality (2.15) for $s(n) = u_{n-1} + u_{n+1} - u_{n-1}u_{n+1}$ as follows; for all $n \geq 3$,

$$s_1(n) < s(n) < s_2(n),$$

where $s_1(n)$ and $s_2(n)$ are combinations of $f(n+1), f(n-1), g(n+1)$, and $g(n-1)$. Therefore, to prove Theorem 2.6 and 2.7, the principal idea behind it is to approximate $s_2(n)$ (cf. (2.14)) and $s_1(n)g(n)^2 - 2g(n) + 1$ (cf. (2.45)) by rational functions in y (=

$y(n) = \pi\sqrt{n}$) (cf. (2.40) and (2.58)). In order to arrive at such estimation to ease the computation, it is necessary to bound the error term $\widetilde{T}(n)$ by $\mu(n)^{-2m}$ for some $m \in \mathbb{Z}_{\geq 1}$ because our estimation turns out to get a suitable polynomial approximation of r, x, z and w (cf. (2.18)) in terms of y. In our case, it is sufficient to consider $m = 3$, as stated in Lemma 2.1.

We denote $\mu(n) = \pi\sqrt{n}$ and define

$$B_1(n) = \frac{e^{\mu(n)}}{8n}\Big(1 - \frac{1}{\mu(n)} - \frac{1}{\mu(n)^6}\Big)$$
$$\text{and} \quad B_2(n) = \frac{e^{\mu(n)}}{8n}\Big(1 - \frac{1}{\mu(n)} + \frac{1}{\mu(n)^6}\Big). \tag{2.1}$$

Lemma 2.1. *For all $n \geq 94$, we have*

$$B_1(n) < \overline{p}(n) < B_2(n). \tag{2.2}$$

Proof. From [17, eqn. (3.5)], it follows that

$$\overline{p}(n) = \frac{e^{\mu(n)}}{8n}\Big(1 - \frac{1}{\mu(n)} + \widetilde{T}(n)\Big) \tag{2.3}$$

where

$$\widetilde{T}(n) = \Big(1 + \frac{1}{\mu(n)}\Big)e^{-2\mu(n)} + \frac{8n}{e^{\mu(n)}}R_2(n,2),$$

and $R_2(n, 2)$ is the error term of (1.4), given in (1.6). By [17, eqn. (3.6)], we have

$$|\widetilde{T}(n)| < 10\ e^{-\frac{1}{2}\mu(n)}. \tag{2.4}$$

Now,

$$10\ e^{-\frac{1}{2}\mu(n)} < \frac{1}{\mu(n)^6} \quad \text{for all } n \geq 275. \tag{2.5}$$

(2.3)-(2.5) altogether imply (2.2) for all $n \geq 275$. We finish the proof by confirming (2.2) by checking numerically in Mathematica for all $94 \leq n \leq 274$. □

Remark 2.2. We note that the upper bound of the absolute value of error term $\widetilde{T}(n)$ can be improved by considering a more generalized version of (2.5) of the following form: there exists $N(m) \in \mathbb{Z}_{\geq 1}$, such that for all $n \geq N(m)$,

$$10\ e^{-\frac{1}{2}\mu(n)} < \frac{1}{\mu(n)^m}. \tag{2.6}$$

Let

$$u_n = \frac{\overline{p}(n-1)\overline{p}(n+1)}{\overline{p}(n)^2} \tag{2.7}$$

and consequently, denote

$$s(n) = u_{n-1} + u_{n+1} - u_{n-1}u_{n+1}. \tag{2.8}$$

Following the notations as given in [14], we set

$$r = \mu(n-2), \quad x = \mu(n-1), \quad y = \mu(n), \quad z = \mu(n+1), \quad w = \mu(n+2), \tag{2.9}$$

and

$$f(n) = e^{x-2y+z}\frac{(x^6-x^5-1)y^{16}(z^6-z^5-1)}{x^8(y^6-y^5+1)^2z^8}, \tag{2.10}$$

$$g(n) = e^{x-2y+z}\frac{(x^6-x^5+1)y^{16}(z^6-z^5+1)}{x^8(y^6-y^5-1)^2z^8}. \tag{2.11}$$

As an immediate consequence of Lemma 2.1, we have the following theorem.

Theorem 2.3. *For all $n \geq 94$,*

$$f(n) < u_n < g(n). \tag{2.12}$$

We begin with the following setup. Define

$$s_1(n) = f(n-1) + f(n+1) - g(n-1)g(n+1), \tag{2.13}$$

and

$$s_2(n) = g(n-1) + g(n+1) - f(n-1)f(n+1). \tag{2.14}$$

As a corollary of Theorem 2.3, we arrive at the following inequality for $s(n)$.

Corollary 2.4. *For $n \geq 94$, we have*

$$s_1(n) < s(n) < s_2(n). \tag{2.15}$$

Now we interpret the Theorem 1.4 in terms of a polynomial expression in $s(n)$ and u_n (cf. (2.7) and (2.8)), given as follows

Theorem 2.5. *For all $n \geq 42$, we have*

$$s(n)u_n^2 - 2u_n + 1 > 0. \tag{2.16}$$

To prove (2.16), first it is required to estimate upper bound of $s(n)$, given by studying $s_2(n)$, as follows;

Theorem 2.6. *For all $n \geq 3$, we have*

$$s_2(n) < 1. \tag{2.17}$$

Proof. For $n \geq 3$, rewriting (2.9), we have

$$r = \sqrt{y^2 - 2\pi^2}, \quad x = \sqrt{y^2 - \pi^2}, \quad z = \sqrt{y^2 + \pi^2}, \quad w = \sqrt{y^2 + 2\pi^2}. \tag{2.18}$$

Expanding $r,\ x,\ z$ and w in terms of y, it follows that

$$\begin{aligned}
r &= y - \frac{\pi^2}{y} - \frac{\pi^4}{2y^3} - \frac{\pi^6}{2y^5} - \frac{5\pi^8}{8y^7} - \frac{7\pi^{10}}{8y^9} - \frac{21\pi^{12}}{16y^{11}} - \frac{33\pi^{14}}{16y^{13}} + O\Big(\frac{1}{y^{15}}\Big),\\
x &= y - \frac{\pi^2}{2y} - \frac{\pi^4}{8y^3} - \frac{\pi^6}{16y^5} - \frac{5\pi^8}{128y^7} - \frac{7\pi^{10}}{256y^9} - \frac{21\pi^{12}}{1024y^{11}} - \frac{33\pi^{14}}{2048y^{13}} + O\Big(\frac{1}{y^{15}}\Big),\\
z &= y + \frac{\pi^2}{2y} - \frac{\pi^4}{8y^3} + \frac{\pi^6}{16y^5} - \frac{5\pi^8}{128y^7} + \frac{7\pi^{10}}{256y^9} - \frac{21\pi^{12}}{1024y^{11}} + \frac{33\pi^{14}}{2048y^{13}} + O\Big(\frac{1}{y^{15}}\Big),\\
w &= y + \frac{\pi^2}{y} - \frac{\pi^4}{2y^3} + \frac{\pi^6}{2y^5} - \frac{5\pi^8}{8y^7} + \frac{7\pi^{10}}{8y^9} - \frac{21\pi^{12}}{16y^{11}} + \frac{33\pi^{14}}{16y^{13}} + O\Big(\frac{1}{y^{15}}\Big).
\end{aligned}$$

It can be easily verified that for all $n \geq 59$,

$$r_1 < r < r_2, \tag{2.19}$$

$$x_1 < x < x_2, \tag{2.20}$$

$$z_1 < z < z_2, \tag{2.21}$$

and

$$w_1 < w < w_2, \tag{2.22}$$

with

$$\begin{aligned}
r_1 &= y - \frac{\pi^2}{y} - \frac{\pi^4}{2y^3} - \frac{\pi^6}{2y^5} - \frac{5\pi^8}{8y^7} - \frac{7\pi^{10}}{8y^9} - \frac{21\pi^{12}}{16y^{11}} - \frac{34\pi^{14}}{16y^{13}},\\
r_2 &= y - \frac{\pi^2}{y} - \frac{\pi^4}{2y^3} - \frac{\pi^6}{2y^5} - \frac{5\pi^8}{8y^7} - \frac{7\pi^{10}}{8y^9} - \frac{21\pi^{12}}{16y^{11}} - \frac{33\pi^{14}}{16y^{13}},\\
x_1 &= y - \frac{\pi^2}{2y} - \frac{\pi^4}{8y^3} - \frac{\pi^6}{16y^5} - \frac{5\pi^8}{128y^7} - \frac{7\pi^{10}}{256y^9} - \frac{21\pi^{12}}{1024y^{11}} - \frac{34\pi^{14}}{2048y^{13}},\\
x_2 &= y - \frac{\pi^2}{2y} - \frac{\pi^4}{8y^3} - \frac{\pi^6}{16y^5} - \frac{5\pi^8}{128y^7} - \frac{7\pi^{10}}{256y^9} - \frac{21\pi^{12}}{1024y^{11}} - \frac{33\pi^{14}}{2048y^{13}},
\end{aligned}$$

$$z_1 = y + \frac{\pi^2}{2y} - \frac{\pi^4}{8y^3} + \frac{\pi^6}{16y^5} - \frac{5\pi^8}{128y^7} + \frac{7\pi^{10}}{256y^9} - \frac{21\pi^{12}}{1024y^{11}},$$
$$z_2 = y + \frac{\pi^2}{2y} - \frac{\pi^4}{8y^3} + \frac{\pi^6}{16y^5} - \frac{5\pi^8}{128y^7} + \frac{7\pi^{10}}{256y^9} - \frac{21\pi^{12}}{1024y^{11}} + \frac{33\pi^{14}}{2048y^{13}},$$
$$w_1 = y + \frac{\pi^2}{y} - \frac{\pi^4}{2y^3} + \frac{\pi^6}{2y^5} - \frac{5\pi^8}{8y^7} + \frac{7\pi^{10}}{8y^9} - \frac{21\pi^{12}}{16y^{11}},$$
$$w_2 = y + \frac{\pi^2}{y} - \frac{\pi^4}{2y^3} + \frac{\pi^6}{2y^5} - \frac{5\pi^8}{8y^7} + \frac{7\pi^{10}}{8y^9} - \frac{21\pi^{12}}{16y^{11}} + \frac{33\pi^{14}}{16y^{13}}.$$

Following the definition of $s_2(n)$ given in (2.14), we see that it suffices to estimate $f(n-1)$, $f(n+1)$, $g(n-1)$ and $g(n+1)$. Now, we observe that each of these four functions consists of two factors, the exponential factor and the rational function in variables x, y and z (cf. (2.10) and (2.11)). This suggests that it is enough to estimate e^{r-2x+y}, e^{y-2z+w}, $h(n-1)$, $h(n+1)$, $q(n-1)$ and $q(n+1)$ individually, where

$$f(n) = e^{x-2y+z}h(n), \quad g(n) = e^{x-2y+z}q(n), \tag{2.23}$$

with

$$h(n) = \frac{(x^6 - x^5 - 1)y^{16}(z^6 - z^5 - 1)}{x^8(y^6 - y^5 + 1)^2 z^8}, \tag{2.24}$$

and

$$q(n) = \frac{(x^6 - x^5 + 1)y^{16}(z^6 - z^5 + 1)}{x^8(y^6 - y^5 - 1)^2 z^8}. \tag{2.25}$$

First, let us consider the exponential factors e^{r-2x+y} and e^{y-2z+w}. By (2.19)-(2.22), for all $n \geq 59$, it follows that

$$e^{r_1-2x_2+y} < e^{r-2x+y} < e^{r_2-2x_1+y}, \tag{2.26}$$
$$e^{y-2z_2+w_1} < e^{y-2z+w} < e^{y-2z_1+w_2}. \tag{2.27}$$

Next, we estimate (2.26) and (2.27) by Taylor expansion of the exponential function in order to get bounds in terms of rational function in y. For convenience, set

$$\Phi(t) = \sum_{i=0}^{6} \frac{t^i}{i!}, \tag{2.28}$$

and

$$\phi(t) = \sum_{i=0}^{7} \frac{t^i}{i!}. \tag{2.29}$$

For $t \in \mathbb{R}_{<0}$,

$$\phi(t) < e^t < \Phi(t). \tag{2.30}$$

We note that

$$r_2 - 2x_1 + y = -\frac{\pi^4\left(1039\pi^{10} + 651\pi^8 y^2 + 420\pi^6 y^4 + 280\pi^4 y^6 + 192\pi^2 y^8 + 128y^{10}\right)}{512y^{13}} < 0$$

and for all $n \geq 2$,

$$y - 2z_1 + w_2 = \frac{\pi^4\left(1056\pi^{10} - 651\pi^8 y^2 + 420\pi^6 y^4 - 280\pi^4 y^6 + 192\pi^2 y^8 - 128y^{10}\right)}{512y^{13}} < 0.$$

Putting (2.30) into (2.26) and (2.27), we get for $n \geq 59$,

$$\phi(r_1 - 2x_2 + y) < e^{r-2x+y} < \Phi(r_2 - 2x_1 + y), \tag{2.31}$$

and

$$\phi(y - 2z_2 + w_1) < e^{y-2z+w} < \Phi(y - 2z_1 + w_2). \tag{2.32}$$

Finally, it remains to estimate $h(n-1)$, $h(n+1)$, $q(n-1)$ and $q(n+1)$. We rewrite these four functions as

$$h(n-1) = \frac{x^{16}\beta(r)\beta(y)}{r^8 y^8 \alpha(x)^2}, \quad h(n+1) = \frac{z^{16}\beta(y)\beta(w)}{w^8 y^8 \alpha(z)^2},$$
$$q(n-1) = \frac{x^{16}\alpha(r)\alpha(y)}{r^8 y^8 \beta(x)^2}, \quad q(n+1) = \frac{z^{16}\alpha(y)\alpha(w)}{w^8 y^8 \beta(z)^2},$$

where

$$\alpha(t) = t^6 - t^5 + 1 \quad \text{and} \quad \beta(t) = t^6 - t^5 - 1. \tag{2.33}$$

Using (2.19)-(2.22), for $n \geq 59$, we put down a list of inequalities as follows

$$\begin{aligned}
r^6 - r_2 r^4 + 1 &< \alpha(r) < r^6 - r_1 r^4 + 1,\\
x^6 - x_2 x^4 + 1 &< \alpha(x) < x^6 - x_1 x^4 + 1,\\
z^6 - z_2 z^4 + 1 &< \alpha(z) < z^6 - z_1 z^4 + 1,\\
w^6 - w_2 w^4 + 1 &< \alpha(w) < w^6 - w_1 w^4 + 1,\\
r^6 - r_2 r^4 - 1 &< \beta(r) < r^6 - r_1 r^4 - 1,\\
x^6 - x_2 x^4 - 1 &< \beta(x) < x^6 - x_1 x^4 - 1,
\end{aligned}$$

$$\begin{aligned}
z^6 - z_2 z^4 - 1 &< \beta(z) < z^6 - z_1 z^4 - 1,\\
w^6 - w_2 w^4 - 1 &< \beta(w) < w^6 - w_1 w^4 - 1,\\
x^{12} - 2x_2x^{10} + x^{10} + 2x^6 - 2x_2x^4 + 1 &< \alpha(x)^2 < x^{12} - 2x_1x^{10} + x^{10} + 2x^6 - 2x_1x^4 + 1,\\
z^{12} - 2z_2z^{10} + z^{10} + 2z^6 - 2z_2z^4 + 1 &< \alpha(z)^2 < z^{12} - 2z_1z^{10} + z^{10} + 2z^6 - 2z_1z^4 + 1,\\
x^{12} - 2x_2x^{10} + x^{10} - 2x^6 + 2x_1x^4 + 1 &< \beta(x)^2 < x^{12} - 2x_1x^{10} + x^{10} - 2x^6 + 2x_2x^4 + 1,\\
z^{12} - 2z_2z^{10} + z^{10} - 2z^6 + 2z_1z^4 + 1 &< \beta(z)^2 < z^{12} - 2z_1z^{10} + z^{10} - 2z^6 + 2z_2z^4 + 1.
\end{aligned} \tag{2.34}$$

By application of the above inequalities, it follows that

$$h(n-1) > \frac{(r^6 - r_2r^4 - 1)x^{16}(y^6 - y^5 - 1)}{r^8y^8(x^{12} - 2x_1x^{10} + x^{10} + 2x^6 - 2x_1x^4 + 1)}, \tag{2.35}$$

$$h(n+1) > \frac{(w^6 - w_2w^4 - 1)z^{16}(y^6 - y^5 - 1)}{w^8y^8(z^{12} - 2z_1z^{10} + z^{10} + 2z^6 - 2z_1z^4 + 1)}, \tag{2.36}$$

$$q(n-1) < \frac{(r^6 - r_1r^4 + 1)x^{16}(y^6 - y^5 + 1)}{r^8y^8(x^{12} - 2x_2x^{10} + x^{10} - 2x^6 + 2x_1x^4 + 1)}, \tag{2.37}$$

$$q(n+1) < \frac{(w^6 - w_1r^4 + 1)z^{16}(y^6 - y^5 + 1)}{r^8y^8(z^{12} - 2z_2z^{10} + z^{10} - 2z^6 + 2z_1z^4 + 1)}. \tag{2.38}$$

Invoking (2.31)-(2.32) and (2.35)-(2.38) into (2.23), for $n \geq 59$, we have

$$\begin{aligned}
g(n-1) < R_1(y) &= \frac{(r^6 - r_1r^4 + 1)x^{16}(y^6 - y^5 + 1)\Phi(r_2 - 2x_1 + y)}{r^8y^8(x^{12} - 2x_2x^{10} + x^{10} - 2x^6 + 2x_1x^4 + 1)},\\
g(n+1) < R_2(y) &= \frac{(w^6 - w_1r^4 + 1)z^{16}(y^6 - y^5 + 1)\Phi(y - 2z_1 + w_2)}{w^8y^8(z^{12} - 2z_2z^{10} + z^{10} - 2z^6 + 2z_1z^4 + 1)},\\
f(n-1) > R_3(y) &= \frac{(r^6 - r_2r^4 - 1)x^{16}(y^6 - y^5 - 1)\phi(r_1 - 2x_2 + y)}{r^8y^8(x^{12} - 2x_1x^{10} + x^{10} + 2x^6 - 2x_1x^4 + 1)},\\
f(n+1) > R_4(y) &= \frac{(w^6 - w_2w^4 - 1)z^{16}(y^6 - y^5 - 1)\phi(y - 2z_2 + w_1)}{w^8y^8(z^{12} - 2z_1z^{10} + z^{10} + 2z^6 - 2z_1z^4 + 1)}.
\end{aligned}$$

By definition of $s_2(n)$ (cf. (2.14)), it suffices to prove that

$$R_1(y) + R_2(y) - R_3(y)R_4(y) - 1 < 0 \tag{2.39}$$

We can reduce $R_1(y) + R_2(y) - R_3(y)R_4(y) - 1$ into a rational function in y; i.e.,

$$R_1(y) + R_2(y) - R_3(y)R_4(y) - 1 = \frac{N_1(y)}{D_1(y)}, \tag{2.40}$$

where $N_1(y)$ and $D_1(y)$ are polynomials in y with respective degree 324 and 330. In order to prove (2.39), it is equivalent to prove $N_1(y)D_1(y) < 0$. We write

$$N_1(y)D_1(y) = \sum_{i=0}^{654} a_i y^i, \tag{2.41}$$

where

$$a_{654} = 2^{348} \cdot 3^8 \cdot 5^4 \cdot 7^4 \cdot (2^8 - \pi^8) < 0.$$

We observe that if a polynomial, say $P(x) = \sum_{i=0}^{m} a_i x^i \in \mathbb{R}[x]$ of degree m with its leading coefficient $a_m \in \mathbb{R}_{<0}$, then $P(x)$ is a decreasing function in x and consequently, $P(x) < 0$ for all $x \geq x_0$ where $x_0 \in \mathbb{R}$. So the only undetermined factor left over is the explicit value of y_0 such that $N_1(y)D_1(y) < 0$ for all $y \geq y_0$, checked with Mathematica that $y_0 = 6$. We conclude the proof by numerical verification that $s_2(n) < 1$ holds for $3 \leq n \leq 59$. $\square$

Next, using the bound of $s_2(n)$ given in Theorem 2.6, we propose an upper bound for $g(n)$ in terms of a function of $s(n)$ that enables us to get into the proof of Theorem 2.5.

Theorem 2.7. *For $t \in (0, 1)$, define*

$$\varphi(t) = \frac{1 - \sqrt{1-t}}{t}. \tag{2.42}$$

Then for all $n \geq 30$, we have

$$g(n) < \varphi(s(n)). \tag{2.43}$$

Proof. We observe that for $t \in (0, 1)$, $\varphi(t)$ is an increasing function in t. From Corollary 2.4 and Theorem 2.6, it suggests that we need to prove for $n \geq 91$,

$$g(n) < \varphi(s_1(n)), \tag{2.44}$$

or equivalently,

$$s_1(n)g(n)^2 - 2g(n) + 1 > 0. \tag{2.45}$$

Recalling the definition of $\alpha(t)$ and $\beta(t)$ (cf. (2.33)), $s_1(n)g(n)^2 - 2g(n) + 1$ can be written in the following form

$$s_1(n)g(n)^2 - 2g(n) + 1 = \frac{-g_1 e^{r+w-2y} + g_2 e^{w+2x-3y} + g_3 - 2g_4 e^{x-2y+z} + g_5 e^{r-3y+2z}}{r^8 w^8 x^{16} z^{16} (x^6 - x^5 - 1)^2 (y^6 - y^5 - 1)^4 (z^6 - z^5 - 1)^2}, \tag{2.46}$$

with

$$g_1 = x^{16}y^{16}z^{16}\alpha(r)\alpha(w)\alpha(x)^2\alpha(y)^2\alpha(z)^2, \tag{2.47}$$

$$g_2 = r^8y^{24}z^{16}\beta(w)\beta(x)^2\alpha(x)^2\beta(y)\beta(z)^2, \tag{2.48}$$

$$g_3 = r^8w^8x^{16}z^{16}\beta(x)^2\beta(y)^4\beta(z)^2, \tag{2.49}$$

$$g_4 = r^8w^8x^8y^{16}z^8\alpha(x)\beta(x)^2\beta(y)^2\alpha(z)\beta(z)^2, \tag{2.50}$$

$$g_5 = w^8x^{16}y^{24}\beta(r)\beta(x)^2\beta(y)\beta(z)^2\alpha(z)^2. \tag{2.51}$$

Since the denominator of (2.46) is a perfect square and hence positive, therefore it is required to prove that

$$G(y) := -g_1e^{r+w-2y} + g_2e^{w+2x-3y} + g_3 - 2g_4e^{x-2y+z} + g_5e^{r-3y+2z} > 0. \tag{2.52}$$

Following a similar method as used in Theorem 2.6, we first estimate the exponential terms in (2.52). It is straightforward to observe that

$$r_2 + w_2 - 2y = -\frac{\pi^4(21\pi^8 + 10\pi^4y^4 + 8y^8)}{8y^{11}} < 0,$$

$$w_1 + 2x_1 - 3y = -\frac{\pi^4(17\pi^{10} + 693\pi^8y^2 - 420\pi^6y^4 + 360\pi^4y^6 - 192\pi^2y^8 + 384y^{10})}{512y^{13}} < 0 \quad \text{for } \ n \geq 1,$$

$$x_2 - 2y + z_2 = -\frac{\pi^4(21\pi^8 + 40\pi^4y^4 + 128y^8)}{512y^{11}} < 0,$$

$$r_1 + 2z_1 - 3y = -\frac{\pi^4(1088\pi^{10} + 693\pi^8y^2 + 420\pi^6y^4 + 360\pi^4y^6 + 192\pi^2y^8 + 384y^{10})}{512y^{13}} < 0.$$

As a consequence, by (2.19)-(2.22) and the monotonicity property of the exponential function, for $n \geq 59$ we have

$$e^{r+w-2y} < e^{r_2+w_2-2y} < \Phi(r_2 + w_2 - 2y), \tag{2.53}$$

$$e^{x+z-2y} < e^{x_2+z_2-2y} < \Phi(x_2 + z_2 - 2y), \tag{2.54}$$

$$e^{w+2x-3y} > e^{w_1+2x_1-3y} > \phi(w_1 + 2x_1 - 3y), \tag{2.55}$$

$$e^{r+2z-3y} > e^{r_1+2z_1-3y} > \phi(r_1 + 2z_1 - 3y). \tag{2.56}$$

Substituting (2.53)-(2.56) into (2.52) implies that for $n \geq 59$,

$$G(y) > -g_1\Phi(r_2+w_2-2y)+g_2\phi(w_1+2x_1-3y)+g_3-2g_4\Phi(x_2+z_2-2y)+g_5\phi(r_1+2z_1-3y). \tag{2.57}$$

The right hand side of the above equation can be simplified further by obtaining its lower bound with the aid of employing (2.34) and (2.18) into the definition of $\{g_\ell\}_{1\leq\ell\leq5}$. More precisely, we have that for $n \geq 59$,

$$G(y) > \frac{N_2(y)}{D_2(y)} := \frac{\sum_{i=0}^{223} b_i y^i}{2^{101} \cdot 3^2 \cdot 5 \cdot 7\ y^{119}}, \tag{2.58}$$

where $b_{223} = 2^{101} \cdot 3^2 \cdot 5 \cdot 7$.
Due to similar remark as before; i.e., if a polynomial, say $P(x) = \sum_{i=0}^{m} a_i x^i \in \mathbb{R}[x]$ of degree m with its leading coefficient $a_m \in \mathbb{R}_{>0}$, then $P(x)$ is an increasing function in x and consequently, $P(x) > 0$ for all $x \geq x_0$ where $x_0 \in \mathbb{R}$. As an immediate consequence, we note that $G(y) > 0$ by verifying that $N_2(y) > 0$ for all $y \geq 5$ or equivalently for $n \geq 3$. It remains to prove (2.43) for $30 \leq n \leq 91$ which is done by numerical checking in Mathematica. □

3. Proof of Theorem 2.5

Proof of Theorem 2.5. By Corollary 2.4 and Theorem 2.6, we have $s(n) < 1$ for $n \geq 91$. Define

$$Q(t) = s(n)t^2 - 2t + 1. \tag{3.1}$$

To establish (2.16), we prove that for $n \geq 91$,

$$Q(u_n) > 0. \tag{3.2}$$

The quadratic equation $Q(t) = 0$ has two solutions, namely

$$t_0 = \frac{1 - \sqrt{1 - s(n)}}{s(n)}, \quad \text{and} \quad t_1 = \frac{1 + \sqrt{1 - s(n)}}{s(n)}.$$

Thus $Q(t) > 0$ when $t < t_0$ or $t > t_1$. From Theorem 2.3 and 2.7, we have that for $n \geq 91$,

$$u_n < \varphi(s(n)) \tag{3.3}$$

Set $t_0 = \varphi(s(n))$ and conclude that (3.2) holds for $n \geq 91$. To confirm (2.5) for $42 \leq n \leq 91$, we can directly verify by checking numerically in Mathematica. □

4. Conclusion

We conclude the paper by undertaking a brief study on totally positive matrices with entries from sequences of overpartitions. Due to Engel [9], we know that for $n \geq 2$,

$$\det\ M_2(\overline{p}(n)) := \det \begin{pmatrix} \overline{p}(n) & \overline{p}(n+1) \\ \overline{p}(n-1) & \overline{p}(n) \end{pmatrix} > 0.$$

Theorem 1.4 states that $\det\ M_3(\overline{p}(n)) > 0$ for $n \geq 42$, more specifically it is worthwhile to observe that for $n \geq 3$, the determinant of each 2×2 minor of the matrix $M_3(\overline{p}(n))$ is nonnegative. This construction leads to the question whether one can always construct a matrix $M_k(\overline{p}(n))$ of order k with positive determinant if we already know its all minors (of lower order) are totally positive. More precisely,

Problem 4.1. For a given $k \in \mathbb{Z}_{\geq 4}$, does there exist a $n(k) \in \mathbb{Z}_{\geq 1}$ such that for $n > n(k)$,

$$\det\,(\overline{p}(n-i+j))_{1\leq i,j\leq k} > 0, \tag{4.1}$$

and if (4.1) holds true, then what is the asymptotic growth of $n(k)$?

An affirmative answer to Problem 4.1 for the case $k = 4$ is recently settled by Wang and Yang [24, Theorem 5.5].

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