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# Log-convexity and the overpartition function

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**Abstract**
Let $\overline{p}(n)$ denote the overpartition function. In this paper, we obtain an inequality for the sequence $\Delta^2 \log \sqrt[n-1]{\overline{p}(n-1)/(n-1)^{\alpha}}$ which states that

$$\log\left(1+\frac{3\pi}{4n^{5/2}}-\frac{11+5\alpha}{n^{11/4}}\right) < \Delta^2 \log \sqrt[n-1]{\overline{p}(n-1)/(n-1)^{\alpha}} < \log\left(1+\frac{3\pi}{4n^{5/2}}\right) \text{ for } n \geq N(\alpha),$$

where $\alpha$ is a non-negative real number, $N(\alpha)$ is a positive integer depending on $\alpha$, and $\Delta$ is the difference operator with respect to $n$. This inequality consequently implies log-convexity of $\left\{\sqrt[n]{\overline{p}(n)/n}\right\}_{n\geq 19}$ and $\left\{\sqrt[n]{\overline{p}(n)}\right\}_{n\geq 4}$. Moreover, it also establishes the asymptotic growth of $\Delta^2 \log \sqrt[n-1]{\overline{p}(n-1)/(n-1)^{\alpha}}$ by showing $\lim_{n\to\infty} \Delta^2 \log \sqrt[n]{\overline{p}(n)/n^{\alpha}} = \frac{3\pi}{4n^{5/2}}$.

**Keywords** Log-convexity · Overpartitions

**Mathematics Subject Classification** Primary 05A20 · 11N37

## 1 Introduction

An overpartition of $n$ is a non-increasing sequence of natural numbers whose sum is $n$ in which the first occurrence of a number may be overlined and $\overline{p}(n)$ denotes the number of overpartitions of $n$. For convenience, define $\overline{p}(0) = 1$. For example, there are 8 overpartitions of 3 enumerated by $3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+$

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$1, \overline{1}+1+1$. Systematic study of overpartition began with the work of Corteel and Lovejoy [4], although it has been studied under different nomenclatures that date back to MacMahon. Analogous to Hardy–Ramanujan–Rademacher formula for partition function (cf. [7], [10]), Zuckerman [13] gave a formula for $\overline{p}(n)$ that reads

$$\overline{p}(n)=\frac{1}{2\pi}\sum_{\substack{k=1\\2\nmid k}}^{\infty}\sqrt{k}\sum_{\substack{h=0\\(h,k)=1}}^{k-1}\frac{\omega(h,k)^2}{\omega(2h,k)}e^{-\frac{2\pi inh}{k}}\frac{d}{dn}\left(\frac{\sinh\frac{\pi\sqrt{n}}{k}}{\sqrt{n}}\right), \tag{1.1}$$

where

$$\omega(h,k)=\exp\left(\pi i\sum_{r=1}^{k-1}\frac{r}{k}\left(\frac{hr}{k}-\left\lfloor\frac{hr}{k}\right\rfloor-\frac{1}{2}\right)\right)$$

for positive integers $h$ and $k$. In somewhat a similar spirit as Lehmer [8] obtained an error bound for the partition function, Engel [6] provided an error term for $\overline{p}(n)$

$$\overline{p}(n)=\frac{1}{2\pi}\sum_{\substack{k=1\\2\nmid k}}^{N}\sqrt{k}\sum_{\substack{h=0\\(h,k)=1}}^{k-1}\frac{\omega(h,k)^2}{\omega(2h,k)}e^{-\frac{2\pi inh}{k}}\frac{d}{dn}\left(\frac{\sinh\frac{\pi\sqrt{n}}{k}}{\sqrt{n}}\right)+R_2(n,N), \tag{1.2}$$

where

$$\left|R_2(n,N)\right|<\frac{N^{5/2}}{\pi n^{3/2}}\sinh\left(\frac{\pi\sqrt{n}}{N}\right). \tag{1.3}$$

A positive sequence $\{a_n\}_{n\geq 0}$ is called log-convex if for $n\geq 1$,

$$a_n^2-a_{n-1}a_{n+1}\leq 0,$$

and it is called log-concave if for $n\geq 1$,

$$a_n^2-a_{n-1}a_{n+1}\geq 0.$$

Engel [6] proved that $\{\overline{p}(n)\}_{n\geq 2}$ is log-concave by using the asymptotic formula (1.2) with $N=2$ followed by (1.3). Prior to Engel's work on overpartitions, log-concavity of partition function $p(n)$ and its associated inequalities has been studied in a broad spectrum, for example see [1], [2], and [5]. Following the same line of studies, Liu and Zhang [9] proved a list of inequalities for overpartition function.

Sun [11] initiated the study on log-convexity problems associated with $p(n)$, later settled by Chen and Zheng [3, Theorem 1.1-1.2]. In a more general setting, Chen and Zheng studied log-convexity of $\{\sqrt[n]{p(n)/n^{\alpha}}\}_{n\geq n(\alpha)}$ (cf. [3, Theorem 1.3]). Moreover, they discovered the asymptotic growth of the sequence $\Delta^2\log\sqrt[n]{p(n)}$ (cf. [3, Theorem 1.4]).

The main objective of this paper is to prove all the theorems [3, Theorem 1.1-1.4] but in the context of overpartitions. Our goal is to obtain a much more general inequality, given in Theorem 1.1, which at once implies [3, Theorem 1.1-1.4] for $\overline{p}(n)$, presented in Corollary 1.2-1.5. More explicitly, in Theorem 1.1, we get a somewhat symmetric upper and lower bound of $\sqrt[n]{\overline{p}(n)/n^{\alpha}}$, as shown in (1.4). We note that the lower bound presented in (1.4) depicts a finer inequality than merely stating $\Delta^2 \log \sqrt[n]{\overline{p}(n)/n^{\alpha}} > 0$ which implies log-convexity. In another direction, we note that (1.4) readily suggests that $\dfrac{3\pi}{4}$ is the best possible constant so as to understand the asymptotic growth of $\Delta^2 \log \sqrt[n]{\overline{p}(n)/n^{\alpha}}$, given in Corollary 1.5.

For $\alpha \in \mathbb{R}_{\geq 0}$, define $r_{\alpha}(n) := \sqrt[n]{\overline{p}(n)/n^{\alpha}}$.

**Theorem 1.1** *Let $\alpha \in \mathbb{R}_{\geq 0}$ and*

$$N(\alpha) := \begin{cases} \max\left\{\left\lceil \dfrac{3490}{\alpha} \right\rceil + 2, \left\lceil \left(\dfrac{4(11+5\alpha)}{3\pi}\right)^4 \right\rceil, 5505\right\} & \text{if } \alpha \in \mathbb{R}_{>0}, \\ 4522 & \text{if } \alpha = 0. \end{cases}$$

*Then for $n \geq N(\alpha)$,*

$$\log\left(1 + \frac{3\pi}{4n^{5/2}} - \frac{11+5\alpha}{n^{11/4}}\right) < \Delta^2 \log r_{\alpha}(n-1) < \log\left(1 + \frac{3\pi}{4n^{5/2}}\right). \tag{1.4}$$

**Corollary 1.2** *The sequence $\left\{\sqrt[n]{\overline{p}(n)/n^{\alpha}}\right\}_{n \geq N(\alpha)}$ is* log-*convex.*

***Proof*** From (1.4), it is immediate that

$$\frac{r_{\alpha}(n+1)r_{\alpha}(n-1)}{r_{\alpha}^2(n)} > 1 + \frac{3\pi}{4n^{5/2}} - \frac{11+5\alpha}{n^{11/4}} \quad \text{for all} \quad n \geq N(\alpha).$$

We finish the proof by observing that

$$1 + \frac{3\pi}{4n^{5/2}} - \frac{11+5\alpha}{n^{11/4}} > 1 \quad \text{for all} \quad n \geq N(\alpha).$$

□

**Corollary 1.3** *The sequences $\left\{\sqrt[n]{\overline{p}(n)/n}\right\}_{n \geq 19}$ and $\left\{\sqrt[n]{\overline{p}(n)}\right\}_{n \geq 4}$ are* log-*convex.*

***Proof*** In order to prove $\left\{\sqrt[n]{\overline{p}(n)/n}\right\}_{n \geq 19}$ and $\left\{\sqrt[n]{\overline{p}(n)}\right\}_{n \geq 4}$ are log-convex, after corollary 1.2, it remains to check numerically for $19 \leq n \leq 5504$ and $4 \leq n \leq 4521$, which is done in 'Mathematica' interface. □

**Corollary 1.4** *For all $n \geq 2$, we have*

$$\frac{\sqrt[n]{\overline{p}(n)}}{\sqrt[n+1]{\overline{p}(n+1)}}\left(1 + \frac{3\pi}{4n^{5/2}}\right) > \frac{\sqrt[n-1]{\overline{p}(n-1)}}{\sqrt[n]{\overline{p}(n)}}. \tag{1.5}$$

***Proof*** It is an immediate implication of (1.4) as it is only left over to verify (1.5) for $2 \leq n \leq 4522$, which we did numerically in 'Mathematica.' □

**Corollary 1.5**

$$\lim_{n\to\infty} n^{5/2} \Delta^2 \log r_\alpha(n) = \frac{3\pi}{4}. \tag{1.6}$$

***Proof*** Multiplying both the sides of (1.4) by $n^{5/2}$ and taking limit as $n$ tends to infinity, we get (1.6). □

## 2 Proof of theorem 1.1

In this section, we give a proof of Theorem 1.1. First, we state the Lemma 2.1 [3, Lemma 2.1] of Chen and Zheng which will be useful in the proofs of Lemmas 2.2-2.4. These lemmas further direct to get upper bound and lower bound of $\Delta^2 \log r_\alpha(n)$, respectively, in Lemma 2.5 and 2.6, finally results (1.4).

**Lemma 2.1** *[3, Lemma 2.1] Suppose $f(x)$ has a continuous second derivative for $x \in [n-1, n+1]$. Then there exists $c \in (n-1, n+1)$ such that*

$$\Delta^2 f(n-1) = f(n+1) + f(n-1) - 2f(n) = f''(c). \tag{2.1}$$

*If $f(x)$ has an increasing second derivative, then*

$$f''(n-1) < \Delta^2 f(n-1) < f''(n+1). \tag{2.2}$$

*Conversely, if $f(x)$ has a decreasing second derivative, then*

$$f''(n+1) < \Delta^2 f(n-1) < f''(n-1). \tag{2.3}$$

We start by laying out a brief outline of Engel's primary set up [6] for proving log-concavity of $\{\overline{p}(n)\}_{n\geq 2}$. Setting $N = 3$ in (1.2), we express $\overline{p}(n)$ as

$$\overline{p}(n) = \overline{T}(n) + \overline{R}(n), \tag{2.4}$$

where

$$\overline{T}(n) = \frac{\overline{c}}{\overline{\mu}(n)^2}\left(1 - \frac{1}{\overline{\mu}(n)}\right)e^{\overline{\mu}(n)}, \tag{2.5}$$

$$\overline{R}(n) = \frac{1}{8n}\left(1 + \frac{1}{\overline{\mu}(n)}\right)e^{-\overline{\mu}(n)} + R_2(n, 3) \tag{2.6}$$

with $\overline{c} = \frac{\pi^2}{8}$ and $\overline{\mu}(n) = \pi\sqrt{n}$. In order to estimate the upper and lower bound of $\Delta^2 \log r_\alpha(n-1)$, it is necessary for us to express $\Delta^2 \log r_\alpha(n-1)$ in the following form

$$\begin{aligned}\Delta^2 \log r_\alpha(n-1) &= \Delta^2 \frac{1}{n-1} \log \overline{p}(n-1) - \alpha\, \Delta^2 \frac{1}{n-1} \log(n-1) \\ &= \Delta^2 \frac{1}{n-1} \log \overline{T}(n-1) + \Delta^2 \frac{1}{n-1} \\ &\quad \log\left(1 + \frac{\overline{R}(n-1)}{\overline{T}(n-1)}\right) - \alpha\, \Delta^2 \frac{1}{n-1} \log(n-1).\end{aligned} \tag{2.7}$$

Define

$$\overline{E}(n-1) = \log\left(1 + \frac{\overline{R}(n-1)}{\overline{T}(n-1)}\right) \tag{2.8}$$

and rewrite (2.7) as

$$\Delta^2 \log r_\alpha(n-1) = \Delta^2 \frac{1}{n-1} \log \overline{T}(n-1) + \Delta^2 \frac{1}{n-1} \overline{E}(n-1)$$
$$-\alpha\, \Delta^2 \frac{1}{n-1} \log(n-1) \tag{2.9}$$

Therefore, in order to estimate $\Delta^2 \log r_\alpha(n-1)$, it is sufficient to estimate each of the three factors, appearing on the right-hand side of (2.9).

**Lemma 2.2** *Let*

$$\overline{G}_1(n) = \frac{3\pi}{4(n+1)^{5/2}} - \frac{5 \log \overline{\mu}(n-1)}{(n-1)^3}, \tag{2.10}$$
$$\overline{G}_2(n) = \frac{3\pi}{4(n-1)^{5/2}} - \frac{3 \log \overline{\mu}(n+1)}{(n+1)^3} + \frac{4}{(n-1)^3}. \tag{2.11}$$

*Then for* $n \geq 2$*, we have*

$$\overline{G}_1(n) < \Delta^2 \frac{1}{n-1} \log \overline{T}(n-1) < \overline{G}_2(n). \tag{2.12}$$

***Proof*** Using the definition of $\overline{T}(n)$ (2.5), we write

$$\Delta^2 \frac{1}{n-1} \log \overline{T}(n-1) = \sum_{i=1}^{4} \Delta^2\, \overline{g}_i(n-1), \tag{2.13}$$

where

$$\overline{g}_1(n) = \frac{\overline{\mu}(n)}{n},$$
$$\overline{g}_2(n) = -\frac{3\log\ \overline{\mu}(n)}{n},$$
$$\overline{g}_3(n) = \frac{\log\ (\overline{\mu}(n)-1)}{n},$$
$$\text{and}\ \ \overline{g}_4(n) = \frac{\log \overline{c}}{n}.$$

It can be easily checked that for $n \geq 3$, $\overline{g}_1^{'''}(n) < 0$, $\overline{g}_2^{'''}(n) > 0$, $\overline{g}_3^{'''}(n) < 0$, and $\overline{g}_4^{'''}(n) < 0$. As a consequence, for $n \geq 3$, $\overline{g}_1^{''}(n)$, $\overline{g}_3^{''}(n)$, and $\overline{g}_4^{''}(n)$ are decreasing, whereas $\overline{g}_2^{''}(n)$ is increasing. Applying Lemma 2.1, we get for $i \in \{1, 3, 4\}$,

$$\overline{g}_i^{''}(n+1) < \Delta^2\, \overline{g}_i(n-1) < \overline{g}_i^{''}(n-1) \tag{2.14}$$

and

$$\overline{g}_2^{''}(n-1) < \Delta^2\, \overline{g}_2(n-1) < \overline{g}_2^{''}(n+1). \tag{2.15}$$

From (2.13) and (2.14)-(2.15), we obtain for all $n \geq 3$,

$$\Delta^2 \frac{1}{n-1} \log \overline{T}(n-1) < \overline{g}_1^{''}(n-1) + \overline{g}_2^{''}(n+1) + \overline{g}_3^{''}(n-1) + \overline{g}_4^{''}(n-1) \tag{2.16}$$

and

$$\Delta^2 \frac{1}{n-1} \log \overline{T}(n-1) > \overline{g}_1^{''}(n+1) + \overline{g}_2^{''}(n-1) + \overline{g}_3^{''}(n+1) + \overline{g}_4^{''}(n+1), \tag{2.17}$$

where

$$\overline{g}_1^{''}(n) = \frac{3\pi}{4n^{5/2}}, \tag{2.18}$$
$$\overline{g}_2^{''}(n) = \frac{9}{2n^3} - \frac{6\log \overline{\mu}(n)}{n^3}, \tag{2.19}$$
$$\overline{g}_3^{''}(n) = \frac{2\log(\overline{\mu}(n)-1)}{n^3} - \frac{5\pi}{4n^{5/2}(\overline{\mu}(n)-1)} - \frac{\pi^2}{4n^2(\overline{\mu}(n)-1)^2}, \tag{2.20}$$
$$\text{and}\ \ \overline{g}_4^{''}(n) = \frac{2\log \overline{c}}{n^3}. \tag{2.21}$$

We first estimate the upper bound of $\Delta^2 \dfrac{1}{n-1} \log \overline{T}(n-1)$ by (2.16) and (2.18)-(2.21).

$$\begin{aligned}\Delta^2 \frac{1}{n-1} \log \overline{T}(n-1) &< \frac{3\pi}{4(n-1)^{5/2}} + \frac{9}{2(n+1)^3} \\ &\quad - \frac{6 \log \overline{\mu}(n+1)}{(n+1)^3} \\ &\quad + \frac{2\log(\overline{\mu}(n-1)-1)}{(n-1)^3} - \frac{5\pi}{4(n-1)^{5/2}(\overline{\mu}(n-1)-1)} \\ &\quad - \frac{\pi^2}{4(n-1)^2(\overline{\mu}(n-1)-1)^2} \\ &\quad + \frac{2\log \overline{c}}{(n-1)^3} \\ &= \frac{3\pi}{4(n-1)^{5/2}} + \overline{U}_1(n) + \overline{U}_2(n), \end{aligned} \tag{2.22}$$

where

$$\overline{U}_1(n) = -\frac{6 \log \overline{\mu}(n+1)}{(n+1)^3} + \frac{2\log(\overline{\mu}(n-1)-1)}{(n-1)^3} \tag{2.23}$$

$$\begin{aligned}\text{and } \overline{U}_2(n) &= \frac{9}{2(n+1)^3} \\ &\quad - \frac{5\pi}{4(n-1)^{5/2}(\overline{\mu}(n-1)-1)} - \frac{\pi^2}{4(n-1)^2(\overline{\mu}(n-1)-1)^2} + \frac{2\log \overline{c}}{(n-1)^3}. \end{aligned} \tag{2.24}$$

It can be easily checked that for all $n \geq 2$,

$$\overline{U}_2(n) < \frac{4}{(n-1)^3}. \tag{2.25}$$

For an upper bound of $\overline{U}_1(n)$, we observe that for all $n \geq 15$,

$$\frac{2}{(n-1)^3} < \frac{3}{(n+1)^3} \quad \text{and} \quad \log(\overline{\mu}(n)-1) < \log \overline{\mu}(n+1), \tag{2.26}$$

that is,

$$\frac{2\log(\overline{\mu}(n-1)-1)}{(n-1)^3} < \frac{3 \log \overline{\mu}(n+1)}{(n+1)^3}. \tag{2.27}$$

Consequently for $n \geq 15$ we get,

$$\overline{U}_1(n) < -\frac{3 \log \overline{\mu}(n+1)}{(n+1)^3}. \tag{2.28}$$

Invoking (2.25) and (2.28) into (2.22), we have for $n \geq 15$,

$$\Delta^2 \frac{1}{n-1} \log \overline{T}(n-1) < \frac{3\pi}{4(n-1)^{5/2}} - \frac{3\log\overline{\mu}(n+1)}{(n+1)^3} + \frac{4}{(n-1)^3} = \overline{G}_2(n). \quad (2.29)$$

For lower bound of $\Delta^2 \frac{1}{n-1} \log \overline{T}(n-1)$, using (2.17) and (2.18)-(2.21) we obtain

$$\begin{aligned}\Delta^2 \frac{1}{n-1} \log \overline{T}(n-1) >& \frac{3\pi}{4(n+1)^{5/2}} + \frac{9}{2(n-1)^3} - \frac{6\log\overline{\mu}(n-1)}{(n-1)^3} \\ &+ \frac{2\log(\overline{\mu}(n+1)-1)}{(n+1)^3} - \frac{5\pi}{4(n+1)^{5/2}(\overline{\mu}(n+1)-1)} \\ &- \frac{\pi^2}{4(n+1)^2(\overline{\mu}(n+1)-1)^2} \\ &+ \frac{2\log\overline{c}}{(n+1)^3} \\ =& \frac{3\pi}{4(n+1)^{5/2}} + \overline{L}_1(n) + \overline{L}_2(n),\end{aligned} \quad (2.30)$$

where

$$\overline{L}_1(n) = -\frac{6\log\overline{\mu}(n-1)}{(n-1)^3} + \frac{2\log(\overline{\mu}(n+1)-1)}{(n+1)^3} \quad (2.31)$$

$$\text{and } \overline{L}_2(n) = \frac{9}{2(n-1)^3} - \frac{5\pi}{4(n+1)^{5/2}(\overline{\mu}(n+1)-1)} - \frac{\pi^2}{4(n+1)^2(\overline{\mu}(n+1)-1)^2} + \frac{2\log\overline{c}}{(n+1)^3}. \quad (2.32)$$

Similarly as before, one can check that for $n \geq 9$,

$$\overline{L}_2(n) > 0 \quad \text{and} \quad \overline{L}_1(n) > -\frac{5\log\overline{\mu}(n-1)}{(n-1)^3}. \quad (2.33)$$

(2.30) and (2.33) yield for $n \geq 9$,

$$\Delta^2 \frac{1}{n-1} \log \overline{T}(n-1) > \frac{3\pi}{4(n+1)^{5/2}} - \frac{5\log\overline{\mu}(n-1)}{(n-1)^3} = \overline{G}_1(n). \quad (2.34)$$

(2.29) and (2.34) together imply (2.12) for $n \geq 15$. We finish the proof by checking (2.12) numerically for $2 \leq n \leq 14$. □

**Lemma 2.3** *For* $n \geq 38$,

$$\left|\Delta^2 \frac{1}{n-1} \overline{E}(n-1)\right| < \frac{5}{n-1} e^{-\frac{\overline{\mu}(n-1)}{12}}. \quad (2.35)$$

***Proof*** Using (2.8), we get for $n \geq 2$,

$$\Delta^2 \frac{1}{n-1}\overline{E}(n-1) = \frac{1}{n+1}\log(1+\overline{e}(n+1))$$
$$-\frac{2}{n}\log(1+\overline{e}(n)) + \frac{1}{n-1}\log(1+\overline{e}(n-1)), \tag{2.36}$$

where

$$\overline{e}(n) = \frac{\overline{R}(n)}{\overline{T}(n)}.$$

Taking absolute value of $\Delta^2 \frac{1}{n-1}\overline{E}(n-1)$ in (2.36), we obtain for all $n \geq 2$,

$$\left|\Delta^2 \frac{1}{n-1}\overline{E}(n-1)\right| \leq \frac{1}{n+1}|\log(1+\overline{e}(n+1))|$$
$$+\frac{2}{n}|\log(1+\overline{e}(n))| + \frac{1}{n-1}|\log(1+\overline{e}(n-1))|. \tag{2.37}$$

Therefore, it is enough to estimate $|\overline{e}(n)|$. Before proceeding to estimate, let us recall the bound of Engel [6](cf. (1.3)) for $N = 3$ that yields for $n \geq 1$,

$$|R_2(n,3)| < \frac{9\sqrt{3}}{2n\,\overline{\mu}(n)} e^{\overline{\mu}(n)/3} \tag{2.38}$$

by making use of the fact that $\sinh(x) < \frac{e^x}{2}$ for $x > 0$. Recalling the definitions in (2.5)-(2.6), we obtain

$$\begin{aligned}
|\overline{e}(n)| &= \left| \frac{\left(1+\frac{1}{\overline{\mu}(n)}\right)}{\left(1-\frac{1}{\overline{\mu}(n)}\right)} e^{-2\overline{\mu}(n)} + \frac{R_2(n,3)}{\frac{1}{8n}\left(1-\frac{1}{\overline{\mu}(n)}\right)e^{\overline{\mu}(n)}} \right| \\
&\leq \frac{\left(1+\frac{1}{\overline{\mu}(n)}\right)}{\left(1-\frac{1}{\overline{\mu}(n)}\right)} e^{-2\overline{\mu}(n)} + \frac{36\sqrt{3}}{\overline{\mu}(n)\left(1-\frac{1}{\overline{\mu}(n)}\right)} e^{-2\overline{\mu}(n)/3} \quad \text{(by (2.38))} \\
&= \frac{e^{-2\overline{\mu}(n)/3}}{\overline{\mu}(n)-1}\left[\left(\overline{\mu}(n)+1\right)e^{-4\overline{\mu}(n)/3} + 36\sqrt{3}\right] \\
&= \frac{e^{-\overline{\mu}(n)/12}}{\overline{\mu}(n)-1}\left[\left(\left(\overline{\mu}(n)+1\right)e^{-4\overline{\mu}(n)/3} + 36\sqrt{3}\right)e^{-\overline{\mu}(n)/2}\right]e^{-\overline{\mu}(n)/12}. \qquad (2.39)
\end{aligned}$$

It can be easily checked that

$$\frac{e^{-\overline{\mu}(n)/12}}{\overline{\mu}(n)-1} < 1 \quad \text{for all } n \geq 1 \tag{2.40}$$

and

$$\Big(\big(\overline{\mu}(n)+1\big)e^{-4\overline{\mu}(n)/3} + 36\sqrt{3}\Big)e^{-\overline{\mu}(n)/2} < 1 \quad \text{for all } n \geq 7. \tag{2.41}$$

Invoking (2.40) and (2.41) into (2.39), we obtain for $n \geq 7$

$$|\overline{e}(n)| < e^{-\overline{\mu}(n)/12} \tag{2.42}$$

and consequently for $n \geq 38$,

$$e^{-\overline{\mu}(n)/12} < \frac{1}{5}. \tag{2.43}$$

Putting together (2.42) and (2.43), we get for all $n \geq 38$,

$$|\overline{e}(n)| < \frac{1}{5}. \tag{2.44}$$

Next we note that for all $n \geq 38$,

$$|\log(1+\overline{e}(n))| \leq \frac{|\overline{e}(n)|}{1-|\overline{e}(n)|} < \frac{5}{4}\,|\overline{e}(n)| \tag{2.45}$$

because of the fact that, for $|x| < 1$,

$$|\log(1+x)| < \frac{|x|}{1-|x|}.$$

From (2.37) and (2.45), we obtain for $n \geq 38$,

$$\Big|\Delta^2\,\frac{1}{n-1}\overline{E}(n-1)\Big| < \frac{5}{4}\Big(\frac{|\overline{e}(n+1)|}{n+1} + 2\frac{|\overline{e}(n)|}{n} + \frac{|\overline{e}(n-1)|}{n-1}\Big). \tag{2.46}$$

Plugging (2.42) into (2.46), we have for $n \geq 38$,

$$\begin{aligned}\Big|\Delta^2\,\frac{1}{n-1}\overline{E}(n-1)\Big| &< \frac{5}{4}\Big(\frac{e^{-\overline{\mu}(n+1)/12}}{n+1} + 2\frac{e^{-\overline{\mu}(n)/12}}{n} + \frac{e^{-\overline{\mu}(n-1)/12}}{n-1}\Big)\\ &< \frac{5}{n-1}e^{-\overline{\mu}(n-1)/12}\end{aligned} \tag{2.47}$$

because the sequence $\left\{\frac{1}{n}\,e^{-\overline{\mu}(n)/12}\right\}_{n\geq 1}$ is decreasing. □

**Lemma 2.4** *For $\alpha \in \mathbb{R}_{>0}$ and $n \geq 7$,*

$$-\frac{2\alpha \log(n-1)}{(n-1)^3} + \frac{3\alpha}{(n-1)^3} < -\alpha\, \Delta^2 \frac{1}{n-1} \log(n-1) < -\frac{2\alpha \log(n+1)}{(n+1)^3} + \frac{3\alpha}{(n+1)^3}. \tag{2.48}$$

***Proof*** We observe that, for $n \geq 7$,

$$\left(-\frac{\log n}{n}\right)''' = -\frac{11}{n^4} + \frac{6 \log n}{n^4} > 0.$$

Setting $f(n) := -\dfrac{\log n}{n}$ and applying Lemma 2.1, we obtain for $n \geq 7$,

$$-\frac{2 \log(n-1)}{(n-1)^3} + \frac{3}{(n-1)^3} < -\Delta^2 \frac{1}{n-1} \log(n-1) < -\frac{2 \log(n+1)}{(n+1)^3} + \frac{3}{(n+1)^3}. \tag{2.49}$$

Since $\alpha$ is a positive real number, from (2.49), we obtain (2.48). □

**Lemma 2.5** *For $\alpha \in \mathbb{R}_{\geq 0}$ and $n \geq 4021$,*

$$\Delta^2 \log r_\alpha(n-1) < \log\left(1 + \frac{3\pi}{4n^{5/2}}\right). \tag{2.50}$$

***Proof*** Using (2.12), (2.35), and (2.48) into (2.9), we obtain for $n \geq 38$,

$$\Delta^2 \log r_\alpha(n-1) < \overline{G}_2(n) + \frac{5}{n-1} e^{-\frac{\overline{\mu}(n-1)}{12}} - \frac{2\alpha \log(n+1)}{(n+1)^3} + \frac{3\alpha}{(n+1)^3}. \tag{2.51}$$

Note that for all $n \geq 4$,

$$-\frac{2\alpha \log(n+1)}{(n+1)^3} + \frac{3\alpha}{(n+1)^3} \leq 0 \tag{2.52}$$

and for $n \geq 4021$,

$$\frac{5}{n-1} e^{-\frac{\overline{\mu}(n-1)}{12}} < \frac{5}{(n-1)^3}. \tag{2.53}$$

Therefore from (2.52)-(2.53), for all $n \geq 4021$, it follows that

$$\Delta^2 \log r_\alpha(n-1) < \frac{3\pi}{4(n-1)^{5/2}} - \frac{3\log\overline{\mu}(n+1)}{(n+1)^3} + \frac{9}{(n-1)^3}. \tag{2.54}$$

Apparently, for all $n \geq 93$,

$$\frac{3\pi}{4(n-1)^{5/2}} - \frac{3\log\overline{\mu}(n+1)}{(n+1)^3} + \frac{9}{(n-1)^3} < \frac{3\pi}{4n^{5/2}} - \frac{9\pi^2}{32n^5}. \tag{2.55}$$

Using the fact that for $x > 0$, $\log(1+x) > x - \dfrac{x^2}{2}$, from (2.54) and (2.55), we finally arrive at

$$\Delta^2 \log r_\alpha(n-1) < \log\left(1 + \frac{3\pi}{4n^{5/2}}\right). \tag{2.56}$$

□

**Lemma 2.6** *For* $\alpha > 0$ *and* $n \geq \max\left\{\left[\frac{3490}{\alpha}\right] + 2, \left\lceil\left(\frac{4(11+5\alpha)}{3\pi}\right)^4\right\rceil, 5505\right\}$,

$$\Delta^2 \log r_\alpha(n-1) > \log\left(1 + \frac{3\pi}{4n^{5/2}} - \frac{11+5\alpha}{n^{11/4}}\right). \tag{2.57}$$

***Proof*** Using (2.12), (2.35), and (2.48) into (2.9), we obtain for $n \geq 38$,

$$\Delta^2 \log r_\alpha(n-1) > \overline{G}_1(n) - \frac{5}{n-1}e^{-\frac{\overline{\mu}(n-1)}{12}} - \frac{2\alpha\log(n-1)}{(n-1)^3} + \frac{3\alpha}{(n-1)^3}. \tag{2.58}$$

It is easy to check that for $n \geq \max\left\{\left[\frac{3490}{\alpha}\right] + 2, 4522\right\} := N_1(\alpha)$,

$$-\frac{5}{n-1}e^{-\frac{\overline{\mu}(n-1)}{12}} + \frac{3\alpha}{(n-1)^3} > \frac{3\alpha}{(n-1)^3} - \frac{10470}{(n-1)^4} > 0. \tag{2.59}$$

Therefore for all $n \geq N_1(\alpha)$,

$$\Delta^2 \log r_\alpha(n-1) > \frac{3\pi}{4(n+1)^{5/2}} - \frac{5\log\overline{\mu}(n-1)}{(n-1)^3} - \frac{2\alpha\log(n-1)}{(n-1)^3}. \tag{2.60}$$

It is immediate that for $n \geq 11$,

$$\log\overline{\mu}(n-1) < \log(n-1) \tag{2.61}$$

and for $n \geq 5505$,

$$\log(n-1) < (n-1)^{1/4}. \tag{2.62}$$

Putting (2.61) and (2.62) into (2.60), we obtain for $n \geq \max\{N_1(\alpha), 5505\}$,

$$\Delta^2 \log r_\alpha(n-1) > \frac{3\pi}{4(n+1)^{5/2}} - \frac{5+2\alpha}{(n-1)^{11/4}}. \tag{2.63}$$

It remains to show that

$$\frac{3\pi}{4(n+1)^{5/2}} - \frac{5+2\alpha}{(n-1)^{11/4}} > \frac{3\pi}{4n^{5/2}} - \frac{11+5\alpha}{n^{11/4}}. \tag{2.64}$$

For $n \geq \max\left\{\left\lceil \left(\frac{15\pi}{8(\alpha+1)}\right)^{4/3} \right\rceil, 5\right\} := N_2(\alpha)$, it follows that

$$\frac{11+5\alpha}{n^{11/4}} - \frac{5+2\alpha}{(n-1)^{11/4}} \underset{n\geq 5}{>} \frac{1+\alpha}{n^{11/4}} > \frac{15\pi}{8n^{7/2}} \underset{n\geq 1}{>} \frac{3\pi}{4}\left(\frac{1}{n^{5/2}} - \frac{1}{(n+1)^{5/2}}\right). \tag{2.65}$$

From (2.63) and (2.64), we obtain for $n \geq \max\left\{N_1(\alpha), N_2(\alpha), 5505\right\}$,

$$\Delta^2 \log r_\alpha(n-1) > \frac{3\pi}{4n^{5/2}} - \frac{11+5\alpha}{n^{11/4}}. \tag{2.66}$$

It is easy to check that for $n \geq \left\lceil \left(\frac{4(11+5\alpha)}{3\pi}\right)^4 \right\rceil := N_3(\alpha)$,

$$\frac{3\pi}{4n^{5/2}} - \frac{11+5\alpha}{n^{11/4}} > 0 \tag{2.67}$$

and using the fact that for $x > 0$, $x > \log(1+x)$, we finally get for $n \geq \max\{N_1(\alpha), N_3(\alpha), 5505\}$ $\Big($since, $N_3(\alpha) > N_2(\alpha)$ for $\alpha > 0\Big)$,

$$\Delta^2 \log r_\alpha(n-1) > \log\left(1 + \frac{3\pi}{4n^{5/2}} - \frac{11+5\alpha}{n^{11/4}}\right). \tag{2.68}$$

□

***Proof of Theorem 1.1*** For $\alpha \in \mathbb{R}_{>0}$, from (2.50) and (2.57) we obtain for all $n \geq \max\left\{\left\lfloor \frac{3490}{\alpha} \right\rfloor + 2, \left\lceil \left(\frac{4(11+5\alpha)}{3\pi}\right)^4 \right\rceil, 5505\right\}$,

$$\log\left(1 + \frac{3\pi}{4n^{5/2}} - \frac{11+5\alpha}{n^{11/4}}\right) < \Delta^2 \log r_\alpha(n-1) < \log\left(1 + \frac{3\pi}{4n^{5/2}}\right). \tag{2.69}$$

For $\alpha = 0$, we have already seen that for $n \geq 4021$,

$$\Delta^2 \log r_\alpha(n-1) < \log\left(1 + \frac{3\pi}{4n^{5/2}}\right). \tag{2.70}$$

For $\alpha = 0$, using (2.12) and (2.35) into (2.9), we get for $n \geq 38$,

$$\Delta^2 \log r_\alpha(n-1) > \overline{G}_1(n) - \frac{5}{n-1} e^{-\frac{\overline{\mu}(n-1)}{12}}. \tag{2.71}$$

Following the same approach, it can be checked that for $n \geq 4522$,

$$-\frac{5}{n-1} e^{-\frac{\overline{\mu}(n-1)}{12}} > -\frac{10470}{(n-1)^4} \tag{2.72}$$

and consequently for $n \geq 476$,

$$\overline{G}_1(n) - \frac{10470}{(n-1)^4} > \frac{3\pi}{4n^{5/2}} - \frac{11}{n^{11/4}} > 0. \tag{2.73}$$

So, for $\alpha = 0$, by (2.71)-(2.73), we obtain for $n \geq 4522$,

$$\Delta^2 \log r_\alpha(n-1) > \log\left(1 + \frac{3\pi}{4n^{5/2}} - \frac{11}{n^{11/4}}\right). \tag{2.74}$$

Putting (2.70) and (2.74), for $n \geq 4522$, it follows that

$$\log\left(1 + \frac{3\pi}{4n^{5/2}} - \frac{11}{n^{11/4}}\right) < \Delta^2 \frac{1}{n-1} \log \overline{p}(n-1) < \log\left(1 + \frac{3\pi}{4n^{5/2}}\right). \tag{2.75}$$

This finishes the proof. □

## 3 Conclusion

We conclude this paper by considering the following problem:

**Problem 3.1** *Let $\alpha$ be a non-negative real number. Then for each $r \geq 1$, does there exists a positive integer $N(r, \alpha)$ so that for all $n \geq N(r, \alpha)$, one can obtain both upper bound and lower bound of $(-1)^r \Delta^r \log r_\alpha(n)$ that finally shows the asymptotic growth of $(-1)^r \Delta^r \log r_\alpha(n)$ as n tends to infinity?*

For $r = 2$, we have already seen that one can estimate $(-1)^r \Delta^r \log r_\alpha(n)$, as given in Theorem 1.4 and its asymptotic growth is reflected in Corollary 1.5.

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