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RESEARCH

Inequalities for higher order differences of the logarithm of the overpartition function and a problem of Wang–Xie–Zhang

Gargi Mukherjee*

*Correspondence:
gargi.mukherjee@dk-compmath.jku.at
Institute for Algebra, Johannes Kepler University, Altenberger Strasse 69, 4040 Linz, Austria

Abstract

Let $\overline{p}(n)$ denote the overpartition function. In this paper, our primary goal is to study the asymptotic behavior of the finite differences of the logarithm of the overpartition function, i.e., $(-1)^{r-1}\Delta^r \log \overline{p}(n)$, by studying the inequality of the following form

$$\log\left(1+\frac{C(r)}{n^{r-1/2}}-\frac{1+C_1(r)}{n^r}\right) < (-1)^{r-1}\Delta^r \log \overline{p}(n) < \log\left(1+\frac{C(r)}{n^{r-1/2}}\right) \text{ for } n \geq N(r),$$

where $C(r)$, $C_1(r)$, and $N(r)$ are computable constants depending on the positive integer r, determined explicitly. This solves a problem posed by Wang, Xie and Zhang in the context of searching for a better lower bound of $(-1)^{r-1}\Delta^r \log \overline{p}(n)$ than 0. By settling the problem, we are able to show that

$$\lim_{n\to\infty} (-1)^{r-1}\Delta^r \log \overline{p}(n) = \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1} n^{\frac{1}{2}-r}.$$

Keywords: Overpartition, log-concavity, Finite difference

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1 Introduction

An overpartition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n in which the first occurrence of a number may be overlined, $\overline{p}(n)$ denotes the number of overpartitions of n, and we define $\overline{p}(0) = 1$. For example, there are 8 overpartitions of 3 enumerated by $3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1$. A thorough study of the overpartition function started with the work of Corteel and Lovejoy [1], although it has been studied under different nomenclature that dates back to MacMahon. Similar to the Hardy-Ramanujan-Rademacher formula for the partition function (cf. [2,3]), Zuckerman's [4] formula for $\overline{p}(n)$ states that

$$\overline{p}(n) = \frac{1}{2\pi}\sum_{\substack{k=1\\2\nmid k}}^{\infty}\sqrt{k}\sum_{\substack{h=0\\(h,k)=1}}^{k-1}\frac{\omega(h,k)^2}{\omega(2h,k)}e^{-\frac{2\pi i n h}{k}}\frac{d}{dn}\left(\frac{\sinh\frac{\pi\sqrt{n}}{k}}{\sqrt{n}}\right), \tag{1.1}$$

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where

$$\omega(h,k)=\exp\left(\pi i\sum_{r=1}^{k-1}\frac{r}{k}\left(\frac{hr}{k}-\left\lfloor\frac{hr}{k}\right\rfloor-\frac{1}{2}\right)\right)$$

for $(h,k)\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 1}$. Engel [5] determined an error term for $\overline{p}(n)$ and found that

$$\overline{p}(n)=\frac{1}{2\pi}\sum_{\substack{k=1\\2\nmid k}}^{N}\sqrt{k}\sum_{\substack{h=0\\(h,k)=1}}^{k-1}\frac{\omega(h,k)^2}{\omega(2h,k)}e^{-\frac{2\pi inh}{k}}\frac{d}{dn}\left(\frac{\sinh\frac{\pi\sqrt{n}}{k}}{\sqrt{n}}\right)+R_2(n,N), \tag{1.2}$$

where

$$\left|R_2(n,N)\right|<\frac{N^{5/2}}{\pi n^{3/2}}\sinh\left(\frac{\pi\sqrt{n}}{N}\right), \tag{1.3}$$

similar to the work done by Lehmer [6] in order to obtain an error bound for the partition function $p(n)$.

A positive sequence $\{a_n\}_{n\geq 0}$ is said to be log-concave (resp. log-convex) if for all $n\geq 1$, $a_n^2\geq a_{n-1}a_{n+1}$ (resp. $a_n^2\leq a_{n-1}a_{n+1}$), and it is said to be strictly log-concave (resp. strictly log-convex) if the inequality is strict.

Using the notations above, Engel's result [5] actually states that $\{\overline{p}(n)\}_{n\geq 1}$ is log-concave. In fact, if one defines $\overline{p}(0):=1$, then $\{\overline{p}(n)\}_{n\geq 0}$ is actually also log-concave. Engel proved that $\{\overline{p}(n)\}_{n\geq 4}$ is strictly log-concave by using the asymptotic formula (1.2) with $N=3$, and the error bound (1.3). Prior to Engel's work on overpartitions, the log-concavity of the partition function $p(n)$ and its associated inequalities has been studied in a wider spectrum, details can be found in [7–9]. On the other hand, Liu and Zhang [10] proved a family of inequalities for the overpartition function. Higher order log-concavity and log-convexity for the overpartition function has been studied in [11,12] respectively.

Chen, Guo and Wang [13] introduced the notion of ratio log-convexity of a sequence and established that ratio log-convexity implies log-convexity under a certain initial condition. A sequence $\{a_n\}_{n\geq k}$ is called ratio log-convex if $\{a_{n+1}/a_n\}_{n\geq k}$ is log-convex or, equivalently, for $n\geq k+1$,

$$\Delta^3\log a_{n-1}=\log a_{n+2}-3\log a_{n+1}+3\log a_n-\log a_{n-1}\geq 0,$$

where Δ be the difference operator defined by $\Delta f(n)=f(n+1)-f(n)$. Chen, Guo, and Wang relates the ratio log-convexity of a sequence, say $\{a_n\}_{n\geq k}$, with strict log-convexity of the associated sequence $\{\sqrt[n]{a_n}\}_{n\geq k}$ stated in the following theorem.

Theorem 1.1 [13, Theorem 3.6] *Let k be a positive integer. If a sequence $\{a_n\}_{n\geq k}$ is ratio* log*-convex and*

$$\frac{\sqrt[k+1]{a_{k+1}}}{\sqrt[k]{a_k}}<\frac{\sqrt[k+2]{a_{k+2}}}{\sqrt[k+1]{a_{k+1}}},$$

then the sequence $\{\sqrt[n]{a_n}\}_{n\geq k}$ is strictly log*-convex.*

Similar to the work done by Chen et al. [8] for $p(n)$, Wang, Xie and Zhang [14] proved the following two theorems.

Theorem 1.2 [14, Theorem 3.1] *For each $r\geq 1$, there exists a positive number $n(r)$ such that for all $n\geq n(r)$,*

$$(-1)^{r-1}\Delta^r\log\overline{p}(n)>0.$$

Theorem 1.3 [14, Theorem 4.1] *For each $r \geq 1$, there exists a positive number $n(r)$ such that for all $n \geq n(r)$,*

$$(-1)^{r-1}\Delta^r \log \overline{p}(n) < \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1}\frac{1}{n^{r-\frac{1}{2}}},$$

where $(\alpha)_r := \alpha \cdot (\alpha+1)\cdots(\alpha+r-1)$.

Remark 1.4 Following Theorem 1.3, we observe that log-concavity and ratio log-convexity for $\overline{p}(n)$ correspond to the cases $r=2$ and $r=3$ respectively.

Wang, Xie, and Zhang raised the following question:

Problem 1.5 *[14, Problem 3.4] Does there exist a positive number A such that*

$$n^{r-1/2}(-1)^{r-1}\Delta^r \log \overline{p}(n) > A,$$

for any $r \geq 1$ and all sufficiently large n?

In other words, their problem reads "*Moreover, we seek a sharp lower bound for* $(-1)^{r-1}\Delta^r \log \overline{p}(n)$".

The main motivation of this paper is to give an affirmative answer to the Problem 1.5 in Theorems 1.6 and 1.8. This in turn clarifies the asymptotic growth of $(-1)^{r-1}\Delta^r \log \overline{p}(n)$, see Corollary 1.9. In Corollaries 1.10 and 1.11, we recover the log-concavity and its (shifted) companion inequality respectively.

Theorem 1.6 *For $n \geq 26$,*

$$\log\left(1+\frac{\pi}{2\sqrt{n}}\right) < \Delta \log \overline{p}(n) < \log\left(1+\frac{\pi}{2\sqrt{n}}+\frac{\pi^2}{40n}\right). \tag{1.4}$$

Definition 1.7 For $r \geq 2$, we define

$$N_0(m) := \begin{cases} 1, & \text{if } m=1, \\ 2m\log m - m\log\log m, & \text{if } m \geq 2, \end{cases} \tag{1.5}$$

$$N_1(r) := \max\left\{85, \left\lceil \frac{4}{\pi^2}N_0^2(2r+2)\right\rceil\right\}, \tag{1.6}$$

$$C(r) := \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1}, \tag{1.7}$$

$$C_1(r) := (r-1)! + 4r^2 C(r), \tag{1.8}$$

$$C_2(r) := \sum_{k=0}^{2r-2}\frac{1}{(k+1)\pi^{k+1}}\left(\frac{k+1}{2}\right)_r \frac{1}{r^k} + \frac{r}{10^r}, \tag{1.9}$$

$$N_2(r) := \left\lceil \left(\frac{1+C_1(r)}{C(r)}\right)^2 \right\rceil, \tag{1.10}$$

$$N_3(r) := \max\left\{N_1(r), 2r^2, \left\lceil \left(\frac{2^{r+1}\left(C_2(r)+1\right)}{(r-1)!}\right)^2 \right\rceil, \left\lceil \sqrt[r-1]{\left(\frac{2^r C(r)^2}{(r-1)!}\right)} \right\rceil\right\}, \tag{1.11}$$

$$\text{and} N(r) := \max\left\{N_2(r), N_3(r)\right\}. \tag{1.12}$$

Theorem 1.8 *For $r \in \mathbb{Z}_{\geq 2}$ and $n \geq N(r)$,*

$$0 < \log\Big(1 + \frac{C(r)}{n^{r-1/2}} - \frac{1 + C_1(r)}{n^r}\Big) < (-1)^{r-1}\Delta^r \log \overline{p}(n) < \log\Big(1 + \frac{C(r)}{n^{r-1/2}}\Big), \tag{1.13}$$

where $C(r)$ and $C_1(r)$ are given in (1.7)–(1.8).

Corollary 1.9 *For $r \in \mathbb{Z}_{\geq 1}$,*

$$\lim_{n\to\infty} n^{r-1/2}(-1)^{r-1}\Delta^r \log \overline{p}(n) = \frac{\pi}{2}\Big(\frac{1}{2}\Big)_{r-1}. \tag{1.14}$$

Proof Multiplying both sides of (1.4) (resp. (1.13)) by $\sqrt{n}$ (resp. by $n^{r-1/2}$) and taking limits as n tends to infinity, we obtain (1.14). □

Corollary 1.10 [5, Theorem 1.2] *For $n \geq 4$, $\overline{p}(n)^2 \geq \overline{p}(n-1)\overline{p}(n+1)$.*

Proof Observe that $N(2) = 344$ and from the lower bound of (1.13), we observe that $\{\overline{p}(n)\}_{n\geq 344}$ is log-concave and for the remaining cases $5 \leq n \leq 343$, we confirm by numerical checking in Mathematica. □

Consider the shifted version of Corollary 1.10, namely, for $n \geq 3, \overline{p}(n+1)^2 \geq \overline{p}(n)\overline{p}(n+2)$. Analogous to [10, Equation (1.6)], we obtain the (shifted) companion inequality in the following form.

Corollary 1.11 *For $n \geq 1$,*

$$\frac{\overline{p}(n)}{\overline{p}(n+1)}\Big(1 + \frac{\pi}{4n^{3/2}}\Big) > \frac{\overline{p}(n+1)}{\overline{p}(n+2)}. \tag{1.15}$$

Proof Using (1.13) with $r = 2$ directly gives (1.15). □

Corollary 1.12 *For $n \geq 18$, $\Delta^3 \log \overline{p}(n-1) > 0$.*

Proof Applying (1.13) with $r = 3$, we observe that for $n \geq 1486 = N(3)$, $\Delta^3 \log \overline{p}(n) > 0$, which is equivalent to say that for $n \geq 1487$, $\Delta^3 \log \overline{p}(n-1) > 0$. For the remaining cases $18 \leq n \leq 1486$, we confirm the inequality $\Delta^3 \log \overline{p}(n-1) > 0$ by numerical checking in Mathematica. □

Define $r(n) := \sqrt[n]{\overline{p}(n)}$.

Corollary 1.13 [11, Corollary 1.3] *For $n \geq 4$, $r(n)^2 < r(n-1)r(n+1)$.*

Proof From Corollary 1.12, we have $\{\overline{p}(n)\}_{n\geq 18}$ is ratio log-convex. Now applying Theorem 1.1 with $k = 18$, and checking numerically

$$\frac{\sqrt[19]{\overline{p}(19)}}{\sqrt[18]{\overline{p}(18)}} < \frac{\sqrt[20]{\overline{p}(20)}}{\sqrt[19]{\overline{p}(19)}},$$

we conclude that $\{r(n)\}_{n\geq 18}$ is strictly log-convex; i.e., for all $n \geq 18$, $r(n)^2 < r(n-1)r(n+1)$, and for the remaining cases $4 \leq n \leq 17$, we confirm by numerical checking in Mathematica. □

This paper is organized as follows. A preliminary setup for decomposing $(-1)^{r-1}\Delta^r \log \overline{p}(n) = H_r + G_r$ (cf. see (2.4), (2.5), and (2.6)), as done in [14], and estimations for both H_r and G_r are given in Sect. 2. Proofs of Theorems 1.6 and 1.8 are given in Sect. 3.

2 preliminary lemmas

Following the notations given in Engel [5] and Wang et al. [14], split $\overline{p}(n)$ as

$$\overline{p}(n) = \widehat{T}(n)\left(1 + \frac{\widehat{R}(n)}{\widehat{T}(n)}\right), \tag{2.1}$$

where

$$\widehat{T}(n) = \frac{1}{8n}\Big(1 - \frac{1}{\widehat{\mu}(n)}\Big)e^{\widehat{\mu}(n)} \tag{2.2}$$

$$\text{and}\widehat{R}(n) = \frac{1}{8n}\Big(1 + \frac{1}{\widehat{\mu}(n)}\Big)e^{-\widehat{\mu}(n)} + R_2(n,3) \tag{2.3}$$

with $\widehat{\mu}(n) = \pi\sqrt{n}$.

Remark 2.1 The splitting for $\overline{p}(n)$ used here is actually slightly different from what is found in [5, 14].

Taking the logarithm on both sides of (2.1) and plugging the definitions from (2.2)–(2.3), we obtain

$$\log\overline{p}(n) = \log\frac{\pi^2}{8} - 3\log\widehat{\mu}(n) + \log(\widehat{\mu}(n)-1) + \widehat{\mu}(n) + \log\left(1 + \frac{\widehat{R}(n)}{\widehat{T}(n)}\right).$$

Therefore,

$$(-1)^{r-1}\Delta^r\log\overline{p}(n) = H_r + G_r, \tag{2.4}$$

where

$$H_r = (-1)^{r-1}\Delta^r(-3\log\widehat{\mu}(n) + \log(\widehat{\mu}(n)-1) + \widehat{\mu}(n)) \tag{2.5}$$

$$G_r = (-1)^{r-1}\Delta^r\log\left(1 + \frac{\widehat{R}(n)}{\widehat{T}(n)}\right). \tag{2.6}$$

Then we have that for $r \geq 1$,

$$H_r - |G_r| \leq (-1)^{r-1}\Delta^r\log\overline{p}(n) \leq H_r + |G_r|. \tag{2.7}$$

To estimate the bounds for $(-1)^{r-1}\Delta^r\log\overline{p}(n)$, we need to establish bounds for H_r and $|G_r|$. Our first goal is to determine a bound for $|G_r|$ for $r \geq 1$ and then we further proceed with H_r but splitting into cases, namely, for $r = 1$ and $r \geq 2$.

Lemma 2.2 [12, Lemma 2.1] *For any integer $m \geq 1$ and $x \geq N_0(m)$,*

$$x^m e^{-x} < 1,$$

where $N_0(m)$ is defined in (1.5).

Recall that $N_1(r) = \max\left\{85, \left\lceil\frac{4}{\pi^2}N_0^2(2r+2)\right\rceil\right\}$ (cf. (1.6)).

Lemma 2.3 *For all $n \geq N_1(r)$ and $r \geq 1$,*

$$|G_r| < \frac{1}{n^{r+1}}. \tag{2.8}$$

Proof Define $\widehat{e}(n) := \frac{\widehat{R}(n)}{\widehat{T}(n)}$. From the definition of $\widehat{R}(n)$ and $\widehat{T}(n)$ (cf. Equations (2.2)–(2.3)), we have

$$\begin{aligned}
|\widehat{e}(n)| &= \frac{|\widehat{R}(n)|}{|\widehat{T}(n)|}\\
&= \left|\frac{\frac{1}{8n}\left(1+\frac{1}{\widehat{\mu}(n)}\right)e^{-\widehat{\mu}(n)} + R_2(n,3)}{\frac{1}{8n}\left(1-\frac{1}{\widehat{\mu}(n)}\right)e^{\widehat{\mu}(n)}}\right|\\
&< \frac{\widehat{\mu}(n)+1}{\widehat{\mu}(n)-1}e^{-2\widehat{\mu}(n)} + \frac{36\sqrt{3}}{\widehat{\mu}(n)-1}e^{-2\widehat{\mu}(n)/3}\\
&\qquad \left(\text{using } N=3 \text{ in (1.3 and } \sinh(x) < \frac{e^x}{2} \text{ for } x>0\right)\\
&= \frac{1}{\widehat{\mu}(n)-1}e^{-\widehat{\mu}(n)/2}\left((\widehat{\mu}(n)+1)e^{-3\widehat{\mu}(n)/2} + 36\sqrt{3}\, e^{-\widehat{\mu}(n)/6}\right).
\end{aligned} \tag{2.9}$$

Since for all $n \geq 85$,

$$(\widehat{\mu}(n)+1)e^{-3\widehat{\mu}(n)/2} + 36\sqrt{3}\, e^{-\widehat{\mu}(n)/6} < \frac{1}{2} \text{ and } \frac{1}{\widehat{\mu}(n)-1} < 1,$$

from (2.9), it follows that for all $n \geq 85$,

$$|\widehat{e}(n)| < \frac{1}{2}\, e^{-\widehat{\mu}(n)/2}. \tag{2.10}$$

Therefore, for all $n \geq 85$,

$$\begin{aligned}
|G_r| &= \left|(-1)^{r-1}\Delta^r \log\left(1+\widehat{e}(n)\right)\right| \quad \text{(by (2.6)}\\
&= \left|\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}\log\left(1+\widehat{e}(n+i)\right)\right|\\
&\leq \sum_{i=0}^{r}\binom{r}{i}\left|\log\left(1+\widehat{e}(n+i)\right)\right|\\
&\leq \sum_{i=0}^{r}\binom{r}{i}\frac{|\widehat{e}(n+i)|}{1-|\widehat{e}(n+i)|} \quad \left(\text{since } |\log(1+x)| \leq \frac{|x|}{1-|x|} \text{ for } |x|<1\right)\\
&\leq 2\sum_{i=0}^{r}\binom{r}{i}|\widehat{e}(n+i)| \quad \left(\text{as } \frac{x}{1-x} \leq 2x \text{ for } 0<x\leq \frac{1}{2}\right)\\
&< \sum_{i=0}^{r}\binom{r}{i}e^{-\widehat{\mu}(n+i)/2} \quad \text{(by (2.10))}\\
&\leq \sum_{i=0}^{r}\binom{r}{i}e^{-\widehat{\mu}(n)/2} \quad \left(\text{since } \{e^{-\widehat{\mu}(n)/2}\}_{n\geq 1} \text{ is a decreasing sequence}\right)\\
&= 2^r e^{-\widehat{\mu}(n)/2}.
\end{aligned} \tag{2.11}$$

Now applying Lemma 2.2 with $m = 2r+2$ and assigning $x \mapsto \frac{\widehat{\mu}(n)}{2}$, it follows that for $n \geq \left\lceil \frac{4}{\pi^2}N_0^2(2r+2)\right\rceil$,

$$e^{-\widehat{\mu}(n)/2} < \left(\frac{2}{\pi}\right)^{2r+2}\frac{1}{n^{r+1}} \implies 2^r e^{-\widehat{\mu}(n)/2} < \left(\frac{2\sqrt{2}}{\pi}\right)^{2r+2}\frac{1}{n^{r+1}} < \frac{1}{n^{r+1}}. \tag{2.12}$$

□

Before we state the bounds for H_r, we recall the following result due to Odlyzko [15] on the relation between the higher order differences of a smooth function and its derivatives. The following proposition can be proved using elementary techniques such as the mean value theorem.

Proposition 2.4 *Let r be a positive integer. Suppose that $f(t)$ is a function with continuous derivatives for $t \geq 1$, and $(-1)^{k-1}f^{(k)}(t) > 0$ for $k \geq 1$. Then for $r \geq 1$, $x \geq 1$,*

$$(-1)^{r-1}f^{(r)}(x+r) \leq (-1)^{r-1}\Delta^r f(x) \leq (-1)^{r-1}f^{(r)}(x).$$

Lemma 2.5 *For all $n \geq 1$,*

$$L^{(1)}(n) \leq H_1 \leq U^{(1)}(n), \tag{2.13}$$

where

$$U^{(1)}(n) = \frac{\pi}{2\sqrt{n}} - \frac{3}{2(n+1)} + \frac{\pi}{2\sqrt{n}(\widehat{\mu}(n)-1)} \tag{2.14}$$

$$\text{and } L^{(1)}(n) = \frac{\pi}{2\sqrt{n+1}} - \frac{3}{2n} + \frac{\pi}{2\sqrt{n+1}(\widehat{\mu}(n+1)-1)}. \tag{2.15}$$

Proof Equation (2.13) follows immediately by applying Proposition 2.4 on each of the factors in H_r being presented in (2.5) for $r = 1$. □

Lemma 2.6 *For $r \geq 2$ and $n \geq 2r^2$,*

$$\frac{C(r)}{n^{r-\frac{1}{2}}} - \frac{C_1(r)}{n^r} < H_r < \frac{C(r)}{n^{r-\frac{1}{2}}} - \frac{(r-1)!}{2^r\ n^r} + \frac{C_2(r)}{n^{r+\frac{1}{2}}}, \tag{2.16}$$

where $C(r)$, $C_1(r)$, and $C_2(r)$ are given by (1.7)–(1.9).

Proof Rewrite (2.5) as

$$H_r = (-1)^{r-1}\Delta^r(\widehat{\mu}(n) - 2\log\widehat{\mu}(n)) - \sum_{k=1}^{\infty}(-1)^{r-1}\Delta^r\Big(\frac{1}{k\widehat{\mu}(n)^k}\Big) \tag{2.17}$$

and applying Proposition 2.4, we get

$$\begin{aligned} &\frac{\pi}{2}\Big(\frac{1}{2}\Big)_{r-1}\frac{1}{(n+r)^{r-\frac{1}{2}}} - \frac{(r-1)!}{n^r} \\ &\quad + \sum_{k=1}^{\infty}\frac{1}{k\pi^k}\Big(\frac{k}{2}\Big)_r\frac{1}{(n+r)^{r+\frac{k}{2}}} \leq H_r \\ &\leq \frac{\pi}{2}\Big(\frac{1}{2}\Big)_{r-1}\frac{1}{n^{r-\frac{1}{2}}} - \frac{(r-1)!}{(n+r)^r} + \sum_{k=1}^{\infty}\frac{1}{k\pi^k}\Big(\frac{k}{2}\Big)_r\frac{1}{n^{r+\frac{k}{2}}}. \end{aligned} \tag{2.18}$$

Since for all positive integers n, r and k,

$$\sum_{k=1}^{\infty}\frac{1}{k\pi^k}\Big(\frac{k}{2}\Big)_r\frac{1}{(n+r)^{r+\frac{k}{2}}} > 0.$$

Therefore,

$$\begin{aligned} &\frac{\pi}{2}\Big(\frac{1}{2}\Big)_{r-1}\frac{1}{(n+r)^{r-\frac{1}{2}}} - \frac{(r-1)!}{n^r} < H_r \\ &\quad \leq \frac{\pi}{2}\Big(\frac{1}{2}\Big)_{r-1}\frac{1}{n^{r-\frac{1}{2}}} - \frac{(r-1)!}{(n+r)^r} + \sum_{k=1}^{\infty}\frac{1}{k\pi^k}\Big(\frac{k}{2}\Big)_r\frac{1}{n^{r+\frac{k}{2}}}. \end{aligned} \tag{2.19}$$

Now we further investigate the lower bound of H_r, given in (2.19).

$$H_r \geq \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1}\frac{1}{(n+r)^{r-\frac{1}{2}}} - \frac{(r-1)!}{n^r}$$
$$= \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1}\frac{1}{n^{r-\frac{1}{2}}}\left(1+\frac{r}{n}\right)^{-r+\frac{1}{2}} - \frac{(r-1)!}{n^r}$$
$$= \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1}\frac{1}{n^{r-\frac{1}{2}}} + \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1}\frac{1}{n^{r-\frac{1}{2}}}\sum_{m=1}^{\infty}\binom{-\frac{2r-1}{2}}{m}\left(\frac{r}{n}\right)^m - \frac{(r-1)!}{n^r}. \tag{2.20}$$

To bound the infinite series in (2.20), we proceed as follows

$$\left|\sum_{m=1}^{\infty}\binom{-\frac{2r-1}{2}}{m}\left(\frac{r}{n}\right)^m\right|$$
$$= \left|\sum_{m=1}^{\infty}\frac{(-1)^m}{4^m}\frac{\binom{2r+2m-2}{r+m-1}\binom{r+m-1}{r-1}}{\binom{2r-2}{r-1}}\left(\frac{r}{n}\right)^m\right|$$
$$\leq \sum_{m=1}^{\infty}\frac{1}{4^m}\frac{\binom{2r+2m-2}{r+m-1}\binom{r+m-1}{r-1}}{\binom{2r-2}{r-1}}\left(\frac{r}{n}\right)^m$$
$$\leq \sum_{m=1}^{\infty}\frac{2\sqrt{r-1}}{\sqrt{\pi(r+m-1)}}\binom{r+m-1}{r-1}\left(\frac{r}{n}\right)^m$$
$$\left(\text{since } \frac{4^k}{2\sqrt{k}} \leq \binom{2k}{k} \leq \frac{4^k}{\sqrt{\pi k}}\ \forall\, k \geq 1\right)$$
$$< \frac{2r}{n}\sum_{m=0}^{\infty}\binom{r+m}{r-1}\left(\frac{r}{n}\right)^m$$
$$\leq \frac{2r}{n}\sum_{m=0}^{\infty} r^{m+1}\left(\frac{r}{n}\right)^m \left(\text{as } \binom{r+m}{r-1} \leq r^{m+1}\ \forall r \geq 1\right)$$
$$= \frac{2r^2}{n}\sum_{m=0}^{\infty}\left(\frac{r^2}{n}\right)^m \leq \frac{4r^2}{n} \quad \text{for all } n \geq 2r^2. \tag{2.21}$$

From (2.20) and (2.21), it follows that for $n \geq 2r^2$,

$$H_r \geq \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1}\frac{1}{n^{r-\frac{1}{2}}} - \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1}\frac{4r^2}{n^{r+\frac{1}{2}}} - \frac{(r-1)!}{n^r}$$
$$> \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1}\frac{1}{n^{r-\frac{1}{2}}} - \left((r-1)! + 2\pi r^2\left(\frac{1}{2}\right)_{r-1}\right)\frac{1}{n^r}. \tag{2.22}$$

This finishes the estimation of the lower bound for H_r.

For the upper bound estimation of H_r, we start with (2.19) in the following way

$$H_r \leq \frac{C(r)}{n^{r-\frac{1}{2}}} - \frac{(r-1)!}{(n+r)^r} + \sum_{k=1}^{\infty}\frac{1}{k\pi^k}\left(\frac{k}{2}\right)_r\frac{1}{n^{r+\frac{k}{2}}}$$
$$< \frac{C(r)}{n^{r-\frac{1}{2}}} - \frac{(r-1)!}{(2n)^r} + \sum_{k=1}^{\infty}\frac{1}{k\pi^k}\left(\frac{k}{2}\right)_r\frac{1}{n^{r+\frac{k}{2}}} \left(\text{since, } \frac{1}{(n+r)^r} > \frac{1}{(2n)^r}\ \forall\, n > r\right)$$
$$= \frac{C(r)}{n^{r-\frac{1}{2}}} - \frac{(r-1)!}{(2n)^r} + \frac{1}{n^{r+\frac{1}{2}}}\sum_{k=0}^{2r-2}\frac{1}{(k+1)\pi^{k+1}}\left(\frac{k+1}{2}\right)_r\frac{1}{\sqrt{n}^k} + \frac{1}{n^{r+\frac{1}{2}}}\sum_{k=2r}^{\infty}\frac{1}{k\pi^k}\left(\frac{k}{2}\right)_r\frac{1}{\sqrt{n}^{k-1}}$$

$$\leq \frac{C(r)}{n^{r-\frac{1}{2}}} - \frac{(r-1)!}{(2n)^r} + \frac{1}{n^{r+\frac{1}{2}}} \underbrace{\sum_{k=0}^{2r-2} \frac{1}{(k+1)\pi^{k+1}} \left(\frac{k+1}{2}\right)_r \frac{1}{r^k}}_{:=\widehat{C_2}(r)} + \frac{r}{n^{r+\frac{1}{2}}} \underbrace{\sum_{k=2r}^{\infty} \frac{1}{k\pi^k} \left(\frac{k}{2}\right)_r \frac{1}{r^k}}_{:=S(r)}$$

$$\left(\text{since } \frac{1}{\sqrt{n}^k} \leq \frac{1}{r^k} \forall\, n \geq r^2\right). \tag{2.23}$$

In order to estimate the infinite series $S(r)$, we need to give an upper bound of $\left(\frac{k}{2}\right)_r$ by rewriting as

$$\left(\frac{k}{2}\right)_r = \left(\frac{k}{2}\right)^r \prod_{i=0}^{r-1}\left(1 + \frac{2i}{k}\right) := \left(\frac{k}{2}\right)^r P(r,k).$$

Now,

$$\log P(r,k) = \sum_{i=0}^{r-1} \log\left(1 + \frac{2i}{k}\right) < \sum_{i=0}^{r-1} \frac{2i}{k} = \frac{r(r-1)}{k} \implies P(r,k) < e^{\frac{r(r-1)}{k}}. \tag{2.24}$$

Using (2.24), we obtain

$$\begin{aligned} S(r) &< \sum_{k=2r}^{\infty} \frac{1}{k\pi^k} \left(\frac{k}{2}\right)^r e^{\frac{r(r-1)}{k}} \frac{1}{r^k} \\ &\leq \frac{e^{\frac{r-1}{2}}}{2^r} \sum_{k=2r}^{\infty} \frac{k^{r-1}}{(\pi r)^k} \left(\text{since } e^{\frac{r(r-1)}{k}} \leq e^{\frac{r-1}{2}} \forall\, k \geq 2r\right). \end{aligned} \tag{2.25}$$

Moreover, $k^{r-1} < r^k$ for all $r \geq 2$ and $k \geq 2r$. To observe this fact, note that $k^{r-1} < r^k$ is equivalent to

$$\frac{r-1}{\log r} < \frac{k}{\log k}. \tag{2.26}$$

Define $f(x) := \dfrac{x}{\log x}$ and observe that $f(x)$ is strictly increasing for all $x > e$. As $k \geq 2r \geq 4 > e$, it follows that $f(k) > f(2r)$ and the fact that $f(2r) > \dfrac{r-1}{\log r}$ for $r \geq 2$, we conclude (2.26).

Applying (2.26) in (2.25), we get

$$S(r) < \frac{e^{\frac{r-1}{2}}}{2^r} \sum_{k=2r}^{\infty} \frac{1}{\pi^k} = \frac{\pi}{\sqrt{e}(\pi-1)} \left(\frac{\sqrt{e}}{2\pi^2}\right)^r < \frac{1}{10^r}. \tag{2.27}$$

Hence, by (2.27) and (2.23), we obtain for all $n \geq r^2$,

$$\begin{aligned} H_r &< \frac{C(r)}{n^{r-\frac{1}{2}}} - \frac{(r-1)!}{2^r\, n^r} + \frac{\widehat{C_2}(r)}{n^{r+\frac{1}{2}}} + \frac{r}{10^r\, n^{r+\frac{1}{2}}} \\ &= \frac{C(r)}{n^{r-\frac{1}{2}}} - \frac{(r-1)!}{2^r\, n^r} + \underbrace{\left(\widehat{C_2}(r) + \frac{r}{10^r}\right)}_{=C_2(r)} \frac{1}{n^{r+\frac{1}{2}}}. \end{aligned} \tag{2.28}$$

□

3 Proof of Theorem 1.6 and 1.8

Proof of Theorem 1.6 Applying (2.13) and (2.8) in (2.7), we have for $n \geq 85 = N_1(1)$,

$$L^{(1)}(n) - \frac{1}{n^2} < \Delta \log \overline{p}(n) < U^{(1)}(n) + \frac{1}{n^2}. \tag{3.1}$$

It is straightforward to show that for $n \geq 457$,

$$-\frac{3}{2(n+1)} + \frac{\pi}{2\sqrt{n}(\widehat{\mu}(n)-1)} + \frac{1}{n^2} < -\frac{\pi^2}{10n} \tag{3.2}$$

and therefore

$$U^{(1)}(n) + \frac{1}{n^2} < \frac{\pi}{2\sqrt{n}} - \frac{\pi^2}{10n}. \tag{3.3}$$

Define $c_n := \frac{\pi}{2\sqrt{n}} - \frac{\pi^2}{10n}$ and $d_n := \frac{\pi}{2\sqrt{n}} + \frac{\pi^2}{40n}$. It can be easily checked that for $n \geq 3$,

$$c_n < d_n - \frac{d_n^2}{2} + \frac{d_n^3}{3} - \frac{d_n^4}{4} < \log(1 + d_n) \tag{3.4}$$

since $\log(1+x) > x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$ for $x > 0$. Invoking (3.3) and (3.4) in (3.1), we get for $n \geq 457$,

$$\Delta \log \overline{p}(n) < \log\Big(1 + \frac{\pi}{2\sqrt{n}} + \frac{\pi^2}{40n}\Big). \tag{3.5}$$

Similarly as before, it can be readily shown that for $n \geq 79$,

$$L^{(1)}(n) - \frac{1}{n^2} > \frac{\pi}{2\sqrt{n}} - \frac{\pi^2}{8n} + \frac{\pi^3}{24n^{3/2}} \tag{3.6}$$

and

$$\frac{\pi}{2\sqrt{n}} - \frac{\pi^2}{8n} + \frac{\pi^3}{24n^{3/2}} > \log\Big(1 + \frac{\pi}{2\sqrt{n}}\Big) \tag{3.7}$$

as $\log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ for $x > 0$. Applying (3.6) and (3.7) into (3.1), it follows that for $n \geq 85$,

$$\Delta \log \overline{p}(n) > \log\Big(1 + \frac{\pi}{2\sqrt{n}}\Big). \tag{3.8}$$

Equations (3.5) and (3.8) conclude the proof of Theorem 1.6 except for $26 \leq n \leq 456$, which we confirm by numerical checking in Mathematica. □

Proof of Theorem 1.8 Applying (2.8) and (2.16) to the lower bound of (2.7), it follows that for $n \geq \max\{N_1(r), 2r^2\}$,

$$(-1)^{r-1}\Delta^r \log \overline{p}(n) > \frac{C(r)}{n^{r-\frac{1}{2}}} - \frac{C_1(r)}{n^r} - \frac{1}{n^{r+1}} > \frac{C(r)}{n^{r-\frac{1}{2}}} - \frac{1+C_1(r)}{n^r}. \tag{3.9}$$

We recall from (1.10) that $N_2(r) = \left\lceil \left(\frac{1+C_1(r)}{C(r)}\right)^2 \right\rceil$. Then for all $n \geq \max\{N_1(r), 2r^2, N_2(r)\}$, it follows that

$$(-1)^{r-1}\Delta^r \log \overline{p}(n) > \frac{C(r)}{n^{r-\frac{1}{2}}} - \frac{1+C_1(r)}{n^r} > \log\left(1 + \frac{C(r)}{n^{r-\frac{1}{2}}} - \frac{1+C_1(r)}{n^r}\right) > 0. \tag{3.10}$$

For the upper bound estimation, putting (2.8) and (2.16) together into the upper bound of (2.7), it follows that for $n \geq \max\{N_1(r), 2r^2\}$,

$$(-1)^{r-1}\Delta^r \log \overline{p}(n) < \frac{C(r)}{n^{r-\frac{1}{2}}} - \frac{(r-1)!}{2^r\, n^r} + \frac{C_2(r)}{n^{r+\frac{1}{2}}} + \frac{1}{n^{r+1}}$$

$$< \frac{C(r)}{n^{r-\frac{1}{2}}} - \frac{(r-1)!}{2^r\, n^r} + \frac{C_2(r)+1}{n^{r+\frac{1}{2}}}. \tag{3.11}$$

Next, our goal is to show for $n \geq N_3(r)$,

$$-\frac{(r-1)!}{2^r\, n^r} + \frac{C_2(r)+1}{n^{r+\frac{1}{2}}} < -\frac{C(r)^2}{2\, n^{2r-1}},$$

which is equivalent to

$$\frac{C(r)^2}{2} < n^{r-1}\left[\frac{(r-1)!}{2^r} - \frac{C_2(r)+1}{\sqrt{n}}\right]. \tag{3.12}$$

Note that for all $n \geq \left\lceil \left(\frac{2^{r+1}\left(C_2(r)+1\right)}{(r-1)!}\right)^2 \right\rceil$, $\frac{(r-1)!}{2^{r+1}} - \frac{C_2(r)+1}{\sqrt{n}} > 0$ and therefore

$$n^{r-1}\left[\frac{(r-1)!}{2^r} - \frac{C_2(r)+1}{\sqrt{n}}\right] = n^{r-1}\left[\frac{(r-1)!}{2^{r+1}} + \frac{(r-1)!}{2^{r+1}} - \frac{C_2(r)+1}{\sqrt{n}}\right] > n^{r-1}\frac{(r-1)!}{2^{r+1}}. \tag{3.13}$$

Hence, to prove (3.12), it is sufficient to prove

$$n^{r-1}\frac{(r-1)!}{2^{r+1}} > \frac{C(r)^2}{2} \text{ which holds for all } n \geq \left\lceil \sqrt[r-1]{\left(\frac{2^r C(r)^2}{(r-1)!}\right)} \right\rceil. \tag{3.14}$$

Recall that

$$N_3(r) = \max\left\{N_1(r), 2r^2, \left\lceil \left(\frac{2^{r+1}\left(C_2(r)+1\right)}{(r-1)!}\right)^2 \right\rceil, \left\lceil \sqrt[r-1]{\left(\frac{2^r C(r)^2}{(r-1)!}\right)} \right\rceil\right\} \text{ (cf. (1.11)).}$$

From (3.11) and (3.12), it follows that for $n \geq N_3(r)$,

$$(-1)^{r-1}\Delta^r \log \overline{p}(n) < \frac{C(r)}{n^{r-\frac{1}{2}}} - \frac{C(r)^2}{2\, n^{2r-1}} < \log\left(1 + \frac{C(r)}{n^{r-1/2}}\right). \tag{3.15}$$

Equations (3.10) and (3.15) together imply that for $n \geq \max\{N_2(r), N_3(r)\} = N(r)$, (1.13) holds. □

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