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Finite difference of the overpartition function

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Let $p(n)$ denote the integer partition function. Good conjectured that $\Delta^r p(n)$ alternates in sign up to a certain value $n = n(r)$, and then it stays positive. Gupta showed that for any given r and sufficiently large n, $\Delta^r p(n) > 0$. Odlyzko proved this conjecture and gave an asymptotic formula for $n(r)$. Then, Almkvist, Knessel and Keller gave many contributions for the exact value of $n(r)$. For the finite difference of $\log p(n)$, DeSalvo and Pak proved that $0 \leq -\triangle^2 \log p(n-1) \leq \log(1+\frac{1}{n})$ and conjectured a sharper upper bound for $-\triangle^2 \log p(n)$. Chen, Wang and Xie proved this conjecture and showed the positivity of $(-1)^{r-1}\triangle^r \log p(n)$, and further gave an upper bound for $(-1)^{r-1}\triangle^r \log p(n)$. As for the overpartition function $\overline{p}(n)$, Engel recently proved that $\overline{p}(n)$ is log-concave for $n \geq 2$, that is, $-\triangle^2 \log \overline{p}(n) \geq 0$ for $n \geq 2$. Motivated by these results, in this paper we will prove the positivity of finite differences of the overpartition function and give an upper bound for $\triangle^r \overline{p}(n)$. Then we show that for any given $r \geq 1$, there exists a positive number $n(r)$ such that $(-1)^{r-1}\triangle^r \log \overline{p}(n) > 0$ for $n > n(r)$, where $\triangle$ is the difference operator with respect to n. Moreover, we give an upper bound for $(-1)^{r-1}\triangle^r \log \overline{p}(n)$.

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1. Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n. In 1918, Hardy and Ramanujan [14] obtained the following asymptotic formula

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2}{3}n}} \quad \text{as } n \to \infty. \tag{1.1}$$

For details, the Hardy–Ramanujan–Rademacher formula for $p(n)$ states that for $N \geq 1$,

$$\begin{aligned} p(n) = {} & \frac{\sqrt{12}}{24n-1} \sum_{k=1}^{N} A_k(n)\sqrt{k} \left[\left(1 - \frac{k}{\mu(n)}\right) e^{\mu(n)/k} + \left(1 + \frac{k}{\mu(n)}\right) e^{-\mu(n)/k} \right] \\ & + R_2(n, N), \end{aligned}$$

where $A_k(n)$ is an arithmetic function, $R_2(n, N)$ is the remainder term and

$$\mu(n) = \frac{\pi}{6}\sqrt{24n-1}, \tag{1.2}$$

see, for example, Hardy and Ramanujan [14], Rademacher [23]. In 1937, Lehmer [18,19] gave the following error bound

$$|R_2(n, N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}} \left[\left(\frac{N}{\mu(n)}\right)^3 \sinh \frac{\mu(n)}{N} + \frac{1}{6} - \left(\frac{N}{\mu(n)}\right)^2 \right],$$

which is valid for all positive integers n and N.

From 1977, many mathematicians began to investigate the finite difference of $p(n)$. Good [12] conjectured that $\Delta^r p(n)$ alternates in sign up to a certain value $n = n(r)$, and then it stays positive, where $\triangle$ is the difference operator denoted by $\triangle f(n) = f(n+1) - f(n)$, and Δ^r is defined recursively in terms of Δ by $\Delta^r = \Delta(\Delta^{r-1})$. Gupta [13] proved that for any given r, $\Delta^r p(n) > 0$ for sufficiently large n. Odlyzko [22] proved this conjecture and gave an asymptotic formula for $n(r)$:

$$n(r) \sim \frac{6}{\pi^2} r^2 \log^2 r \quad \text{as } r \to \infty. \tag{1.3}$$

Then, Knessl and Keller [16,17] used WKB method to obtain an approximation $n(r)'$ for $n(r)$ for which $|n(r)' - n(r)| \leq 2$ up to $r = 75$. Moreover, Almkvist [2,3] proved that $n(r)$ satisfies certain equations.

Recently, using the Hardy–Ramanujan–Rademacher formula and Lehmer's error bound, DeSalvo and Pak [9] proved the following inequalities conjectured by Chen [4].

Theorem 1.1. *For $n \geq 26$,*

$$\frac{p(n)}{p(n+1)} > \frac{p(n-1)}{p(n)},$$

and for $n \geq 2$,

$$\frac{p(n-1)}{p(n)}\left(1+\frac{1}{n}\right) > \frac{p(n)}{p(n+1)}.$$

They also proposed the following conjecture.

Conjecture 1.2. *For $n \geq 45$, we have*

$$\frac{p(n-1)}{p(n)}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{p(n)}{p(n+1)}.$$

Chen, Wang and Xie [6] proved this conjecture and further obtained the following theorem analogous to the positivity of $\triangle^r p(n)$ obtained by Gupta [13].

Theorem 1.3. *For each $r \geq 1$, there exists a positive integer $n(r)$ such that for $n \geq n(r)$,*

$$(-1)^{r-1}\triangle^r \log p(n) > 0. \tag{1.4}$$

They also gave the following upper bound for $(-1)^{r-1}\triangle^r \log p(n)$.

Theorem 1.4. *For each $r \geq 1$, there exists a positive integer $n(r)$ such that for $n \geq n(r)$,*

$$(-1)^{r-1}\triangle^r \log p(n) < \log\left(1+\frac{\sqrt{6}\pi}{6}\left(\frac{1}{2}\right)_{r-1}\frac{1}{(n+1)^{r-\frac{1}{2}}}\right), \tag{1.5}$$

where $(\frac{1}{2})_{r-1}$ is the rising factorial, namely, $(\frac{1}{2})_{r-1} = \frac{1}{2}(\frac{1}{2}+1)\cdots(\frac{1}{2}+k-1)$ for $k \geq 1$.

In this paper, we extend these results to overpartitions. An overpartition of n is a partition of n for which the first occurrence (equivalently, the last occurrence) of a number may be overlined. For example, the eight overpartitions of 3 are $3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1$. Overpartitions play an important role in hypergeometric series identities [20,21], supersymmetric functions and mathematical physics [10], representation theory and Lie algebras [15].

Let $\overline{p}(n)$ denote the number of overpartitions of n. Hardy and Ramanujan [14] stated that

$$\overline{p}(n) = \frac{1}{4\pi}\frac{d}{dn}\frac{e^{\pi\sqrt{n}}}{\sqrt{n}} + \frac{\sqrt{3}}{2\pi}\cos\left(\frac{2}{3}n\pi - \frac{1}{6}\pi\right)\frac{d}{dn}\left(\frac{e^{\pi\sqrt{n}/3}}{\sqrt{n}}\right) + \cdots + O\left(n^{-1/4}\right). \tag{1.6}$$

Zuckerman [25] gave the following Rademacher-type convergent series

$$\overline{p}(n)=\frac{1}{2\pi}\sum_{\substack{k\geq 1\\ 2\nmid k}}\sqrt{k}\sum_{\substack{0\leq h<k\\ (h,k)=1}}\frac{\omega(h,k)^2}{\omega(2h,k)}e^{-2\pi ihn/k}\frac{d}{dn}\left(\frac{\sinh(\pi\sqrt{n}/k)}{\sqrt{n}}\right), \tag{1.7}$$

where

$$\omega(h,k)=\exp\left(\pi i\sum_{r=1}^{k-1}\frac{r}{k}\left(\frac{hr}{k}-\left\lfloor\frac{hr}{k}\right\rfloor-\frac{1}{2}\right)\right). \tag{1.8}$$

Recently, Engel [11] split $\overline{p}(n)$ into two parts for any integer $N\geq 1$

$$\overline{p}(n)=\frac{1}{2\pi}\sum_{\substack{k\geq 1\\ 2\nmid k}}\sqrt{k}\hat{A}_k(n)\frac{d}{dn}\left(\frac{\sinh(\hat{\mu}/k)}{\sqrt{n}}\right) \tag{1.9}$$

$$=\frac{1}{2\pi}\sum_{\substack{k\geq 1\\ 2\nmid k}}^{N}\sqrt{k}\hat{A}_k(n)\frac{d}{dn}\left(\frac{\sinh(\hat{\mu}/k)}{\sqrt{n}}\right)+R_2(n,N), \tag{1.10}$$

and gave the following error bound

$$|R_2(n,N)|\leq\frac{N^{5/2}}{n\hat{\mu}}\sinh\left(\frac{\hat{\mu}}{N}\right), \tag{1.11}$$

where

$$\hat{\mu}=\hat{\mu}(n)=\pi\sqrt{n},$$

$$\hat{A}_k(n)=\sum_{\substack{0\leq h<k\\ (h,k)=1}}\frac{\omega(h,k)^2}{\omega(2h,k)}e^{-2\pi ihn/k}. \tag{1.12}$$

Using the error bound (1.11), Engel [11] deduced the log-concavity of $\overline{p}(n)$:

Theorem 1.5. *The function $\overline{p}(n)$ is log-concave for $n\geq 2$.*

Motivated by these results, we shall give an exact formula for $\triangle^r\overline{p}(n)$ and show that for any given r, there exists $n(r)$ such that $\triangle^r\overline{p}(n)$ is positive for $n>n(r)$. We also give an upper bound for $\triangle^r\overline{p}(n)$ in Section 2. In Section 3, we shall show the positivity of $(-1)^{r-1}\triangle^r\log\overline{p}(n)$ for sufficiently large n. At last, we also give an upper bound for $(-1)^{r-1}\triangle^r\log\overline{p}(n)$.

2. The positivity of $\triangle^r\overline{p}(n)$

In this section, we will prove that for any given $r \geq 1$, there is an integer $n(r)$ such that for $n \geq n(r)$, $\triangle^r\overline{p}(n) > 0$, where $\triangle$ is the difference operator denoted by $\triangle f(n) = f(n+1) - f(n)$, and Δ^r is defined recursively in terms of Δ by $\Delta^r = \Delta(\Delta^{r-1})$. This is analogous to the positivity of finite differences of the partition function, which has been extensively studied by Good [12], Gupta [13], Odlyzko [22], Knessl and Keller [16,17], and Almkvist [2,3].

Theorem 2.1. *For each $r \geq 1$, there exists a positive integer $n(r)$ such that for $n \geq n(r)$,*

$$\triangle^r\overline{p}(n) > 0.$$

To prove the above theorem, we introduce an important theorem given by Almkvist [2]. First, let us introduce some notations. Let

$$L_\nu(x) = \sum_{m\geq 0} \frac{x^m}{m!\Gamma(m+\nu+1)}.$$

Consider the function

$$F(x) = \sum_{n=1}^{\infty} a(n)x^n,$$

where $a(n)$ can be represented by the following form

$$a(n) = \sum_{k\geq 1} \sum_{(h,k)=1} u(h,k)e^{-2\pi i hn/k} L_\nu(d_k(n+\alpha)),$$

where ν and α are constants depending on $F(x)$, and $u(h,k)$ and d_k are complex numbers. We assume that $a(0) = 1$, $a(n) = 0$ for $n < 0$.

Denote the function $g(x)$ be the generating function of $b(n)$, namely,

$$g(x) = \sum_{n\geq 0} b(n)x^n.$$

Define

$$S(x) = g(x)/F(x) = \sum_{n\geq 0} e(n)x^n,$$

and

$$S_n(x) = \sum_{m=0}^{n} e(m)x^m.$$

Almkvist [2] gave the following theorem.

Theorem 2.2. *Let* $F(x) = \sum_0^\infty a(n)x^n$ *be a function satisfying above conditions. Then we have*

$$b(n) = \sum_{k=1}^{\infty} \sum_{(h,k)=1} \omega(h,k)e^{-2\pi ihn/k} S_n \left(e^{-(D-2\pi ih/k)}\right) L_\nu(d_k(n+\alpha)),$$

where D *is the differential operator* d/dn.

Proof of Theorem 2.1. It is known that

$$L_{3/2}(x) = \frac{1}{\sqrt{\pi}} \frac{d}{dx} \left(\frac{\sinh(2\sqrt{x})}{\sqrt{x}} \right),$$

see Abramowitz and Stegun [1] or Chen, Wang and Xie [6]. Thus, we have

$$\frac{d}{dx} \left(\frac{\sinh(\pi\sqrt{x}/k)}{\sqrt{x}} \right) = \frac{\sqrt{\pi}}{8} \left(\frac{\pi}{k}\right)^3 L_{3/2}\left(\frac{\pi^2}{4k^2}x\right). \tag{2.1}$$

Applying (2.1) to (1.7), we obtain that

$$\overline{p}(n) = \frac{1}{16}\pi^{\frac{5}{2}} \sum_{\substack{k\geq 1\\ 2\nmid k}} \sum_{\substack{0\leq h<k\\ (h,k)=1}} \frac{\omega(h,k)^2}{\omega(2h,k)} e^{-2\pi ihn/k} k^{-\frac{5}{2}} L_{3/2}\left(\frac{\pi^2}{4k^2}n\right).$$

Let $F(x)$ denote the generating function of overpartition $\overline{p}(n)$

$$F(x) = \sum_{n\geq 1} \overline{p}(n)x^n.$$

It is easy to see that

$$F(x) = \prod_{n\geq 1} \frac{1+x^n}{1-x^n},$$

see Corteel and Lovejoy [8].

Based on the above formula, we can deduce that the generating function of $\triangle^r\overline{p}(n-r)$. We claim that the generating function $g(n)$ of $\triangle^r\overline{p}(n-r)$ has the following form:

$$g(x) = \sum_{n\geq 1} \triangle^r\overline{p}(n-r)x^n = (1-x)^r \prod_{n\geq 1} \frac{1+x^n}{1-x^n}. \tag{2.2}$$

We prove it by induction on r. We assume that $\overline{p}(n) = 0$ for $n \leq 0$. For $r = 1$, it can be checked that

$$\begin{aligned}\sum_{n\geq 1}\triangle\overline{p}(n-1)x^n &= \sum_{n\geq 1}(\overline{p}(n)-\overline{p}(n-1))x^n \\ &= \sum_{n\geq 1}\overline{p}(n)x^n - x\sum_{n\geq 0}\overline{p}(n-1)x^{n-1} \\ &= (1-x)\prod_{n\geq 1}\frac{1+x^n}{1-x^n}.\end{aligned}$$

Suppose that (2.2) is true for $r=k$. Then

$$\begin{aligned}\sum_{n\geq 1}\triangle^{k+1}\overline{p}(n-k-1)x^n &= \sum_{n\geq 0}\left(\triangle^k\overline{p}(n-k)-\triangle^k\overline{p}(n-k-1)\right)x^n \\ &= \sum_{n\geq 1}\triangle^k\overline{p}(n-k)x^n - x\sum_{n\geq 1}\triangle^k\overline{p}(n-k-1)x^{n-1} \\ &= (1-x)^k\prod_{n\geq 1}\frac{1+x^n}{1-x^n} - (1-x)^k x\prod_{n\geq 1}\frac{1+x^n}{1-x^n} \\ &= (1-x)^{k+1}\prod_{n\geq 1}\frac{1+x^n}{1-x^n}.\end{aligned}$$

So (2.2) is true for $r=k+1$. This shows that (2.2) is true for all positive integers n.

Then, we have

$$S(x) = \frac{g(x)}{F(x)} = (1-x)^r.$$

By the definition of $S_n(x)$, we find that for $n \geq r$,

$$S_n(x) = (1-x)^r.$$

Hence, by Theorem 2.2, we find that for $n \geq r$,

$$\triangle^r\overline{p}(n-r) = \frac{1}{16}\pi^{\frac{5}{2}}\sum_{\substack{k\geq 1\\ 2\nmid k}}\sum_{\substack{0\leq h<k\\ (h,k)=1}}\frac{\omega(h,k)^2}{\omega(2h,k)}e^{-2\pi ihn/k}k^{-\frac{5}{2}}S_n\left(e^{-(D-2\pi ih/k)}\right)L_{3/2}\left(\frac{\pi^2}{4k^2}n\right).$$

Since

$$\sinh\frac{D}{2} = \sum_{n=0}^{\infty}\frac{(\frac{D}{2})^{2n+1}}{(2n+1)!} = \frac{D}{2}\left(1+\frac{D^2}{24}+\cdots\right),$$

we deduce that for $n \geq r$,

$$S_n(e^{-D}) = (1-e^{-D})^r = 2^r e^{-rD/2}\left(\sinh\frac{D}{2}\right)^r$$
$$= e^{-rD/2}D^r\left(1+\frac{r}{24}D^2+\cdots\right).$$

It follows that for $n \geq r$,

$$\triangle^r\overline{p}(n-r) = \frac{1}{16}\pi^{\frac{5}{2}}\Bigg(e^{-rD/2}D^r\left(1+\frac{r}{24}D^2+\cdots\right) + \sum_{\substack{k\geq 3\\ 2\nmid k}}\sum_{\substack{0\leq h<k\\ (h,k)=1}}\frac{\omega(h,k)^2}{\omega(2h,k)}e^{-2\pi ihn/k}k^{-\frac{5}{2}}S_n\left(e^{-(D-2\pi ih/k)}\right)\Bigg)L_{3/2}\left(\frac{\pi^2}{4}n\right). \tag{2.3}$$

By the definition of function $L_\nu(n)$, it is easily verified that

$$DL_\nu(n) = L_{\nu+1}(n),$$

so

$$D^r L_{3/2}\left(\frac{\pi^2}{4}n\right) = \left(\frac{\pi^2}{4}\right)^r L_{r+3/2}\left(\frac{\pi^2}{4}n\right). \tag{2.4}$$

And by Taylor's theorem, we have that

$$e^{-rD/2}L_\nu\left(\frac{\pi^2}{4}n\right) = L_\nu\left(\frac{\pi^2}{4}\left(n-\frac{r}{2}\right)\right). \tag{2.5}$$

Applying (2.4) and (2.5) to (2.3) and replacing $n-r$ with n, we obtain that

$$\triangle^r\overline{p}(n) = \frac{1}{16}\pi^{\frac{5}{2}}\left(\left(\frac{\pi^2}{4}\right)^r L_{r+3/2}\left(\frac{\pi^2}{4}\left(n+\frac{r}{2}\right)\right) + \frac{r}{24}\left(\frac{\pi^2}{4}\right)^{r+2}L_{r+7/2}\left(\frac{\pi^2}{4}\left(n+\frac{r}{2}\right)\right)+\cdots\right) + \frac{1}{16}\pi^{\frac{5}{2}}\sum_{\substack{k\geq 3\\ 2\nmid k}}\sum_{\substack{0\leq h<k\\ (h,k)=1}}\frac{\omega(h,k)^2}{\omega(2h,k)}e^{-2\pi ih(n+r)/k}k^{-\frac{5}{2}}S_n\left(e^{-(D-2\pi ih/k)}\right)L_{3/2}\left(\frac{\pi^2}{4}(n+r)\right). \tag{2.6}$$

On the other hand, applying (2.1) to (1.9), we have

$$\overline{p}(n) = \frac{1}{16}\pi^{\frac{5}{2}}\sum_{\substack{k\geq 1\\ 2\nmid k}}\hat{A}_k(n)k^{-\frac{5}{2}}L_{3/2}\left(\frac{\pi^2}{4k^2}n\right). \tag{2.7}$$

Denote the kth term in (2.7) by $f_k(n)$, namely,

$$f_k(n) = \frac{1}{16}\pi^{\frac{5}{2}}\hat{A}_k(n)k^{-\frac{5}{2}}L_{3/2}\left(\frac{\pi^2}{4k^2}n\right). \tag{2.8}$$

Now we estimate the rth difference of $f_k(n)$. First, from the proof of Almkvist's theorem, one can get that $\triangle^r f_1(n)$ is the first sum in (2.6), that is,

$$\triangle^r f_1(n) = \frac{1}{16}\pi^{\frac{5}{2}}\left(\left(\frac{\pi^2}{4}\right)^r L_{r+3/2}\left(\frac{\pi^2}{4}\left(n+\frac{r}{2}\right)\right) + \frac{r}{24}\left(\frac{\pi^2}{4}\right)^{r+2} L_{r+7/2}\left(\frac{\pi^2}{4}\left(n+\frac{r}{2}\right)\right) + \cdots\right).$$

Then, we can estimate $\triangle^r f_1(n)$ as follow

$$\triangle^r f_1(n) \geq \frac{1}{16}\pi^{\frac{5}{2}}\left(\frac{\pi^2}{4}\right)^r L_{r+3/2}\left(\frac{\pi^2}{4}\left(n+\frac{r}{2}\right)\right). \tag{2.9}$$

Now we turn to give a lower bound for $|\triangle^r f_k(n)|$ for $k \geq 3$. Since for any function $f(n)$,

$$\triangle^r f(n) = \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k} f(n+k), \tag{2.10}$$

thus, by (2.8) we have that

$$|\triangle^r f_k(n)| = \frac{1}{16}\pi^{\frac{5}{2}} k^{-\frac{5}{2}}\left|\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}\hat{A}_k(n+i)L_{3/2}\left(\frac{\pi^2}{4k^2}(n+i)\right)\right|.$$

It is easily seen that $L_{3/2}(x)$ increases with x, and $|\hat{A}_k(n)| \leq k$. So

$$\begin{aligned} |\triangle^r f_k(n)| &\leq \frac{1}{16}\pi^{\frac{5}{2}} k^{-\frac{5}{2}} \cdot 2^r \cdot k L_{3/2}\left(\frac{\pi^2}{4k^2}(n+r)\right) \\ &\leq \frac{1}{16}\pi^{\frac{5}{2}} 2^r \cdot k^{-\frac{3}{2}} L_{3/2}\left(\frac{\pi^2}{36}(n+r)\right). \end{aligned} \tag{2.11}$$

Thus, summing $|\triangle^r f_k(n)|$ over all k is odd and $k \geq 3$, we arrive at

$$\sum_{k\geq 3, 2\nmid k} |\triangle^r f_k(n)| \leq \frac{1}{16}\pi^{\frac{5}{2}} 2^r \zeta(3/2) L_{3/2}\left(\frac{\pi^2}{36}(n+r)\right), \tag{2.12}$$

where $\zeta(x)$ is the Riemann zeta function.

Comparing (2.12) with (2.9), we claim that there exists a positive integer $n_1(r)$ such that for $n \geq n_1(r)$,

$$\triangle^r f_1(n) > \sum_{k\geq 3, 2\nmid k} |\triangle^r f_k(n)|. \tag{2.13}$$

For convenience, we denote the right hand side of (2.9) and (2.12) by $g(n)$ and $h(n)$, respectively. That is,

$$f(n) = \frac{1}{16}\pi^{\frac{5}{2}}\left(\frac{\pi^2}{4}\right)^r L_{r+3/2}\left(\frac{\pi^2}{4}\left(n+\frac{r}{2}\right)\right), \tag{2.14}$$

$$h(n) = \frac{1}{16}\pi^{\frac{5}{2}}2^r\zeta(3/2)L_{3/2}\left(\frac{\pi^2}{36}(n+r)\right). \tag{2.15}$$

By the definition of $L_\nu(x)$, we have that for any given r,

$$L_{r+3/2}\left(\frac{\pi^2}{4}\left(n+\frac{r}{2}\right)\right) = \sum_{m\geq 0}\frac{\pi^{2m}}{4^m m!\Gamma(m+r+5/2)}\left(n+\frac{r}{2}\right)^m, \tag{2.16}$$

$$L_{3/2}\left(\frac{\pi^2}{36}(n+r)\right) = \sum_{m\geq 0}\frac{\pi^{2m}}{4^m m!\Gamma(m+5/2)}\left(\frac{n}{9}+\frac{r}{9}\right)^m. \tag{2.17}$$

It is easily seen that $L_{r+3/2}\left(\frac{\pi^2}{4}\left(n+\frac{r}{2}\right)\right)$ and $L_{3/2}\left(\frac{\pi^2}{36}(n+r)\right)$ both increase with n. Thus, by the definition of $g(n)$ and $h(n)$, we get that both of them increase with n. For large n, $g(n)$ and $h(n)$ are dominated by $(n+r/2)^m/\Gamma(m+r+5/2)$ and $(n/9+r/9)^m/\Gamma(m+5/2)$, respectively, and $(n+r/2)^m/\Gamma(m+r+5/2) > (n/9+r/9)^m/\Gamma(m+5/2)$ for large n. Thus $g(n)$ is larger than $h(n)$ for sufficiently large n, that is, for $n \geq n_1(r)$,

$$\triangle^r f_1(n) \geq \sum_{k\geq 3, 2\nmid k}|\triangle^r f_k(n)|, \tag{2.18}$$

where $n_1(r)$ may be taken to be the solution of the equation $g(n) = h(n)$, i.e., the solution of

$$2^r\zeta(3/2)L_{3/2}\left(\frac{\pi^2}{36}(n+r)\right) = \left(\frac{\pi^2}{4}\right)^r L_{r+3/2}\left(\frac{\pi^2}{4}\left(n+\frac{r}{2}\right)\right). \tag{2.19}$$

One can obtain an approximate solution of the above equation by using Newton–Raphson method.

Summing up, for each $r \geq 1$, let $n(r) = \max\{r, n_1(r)+1\}$, we conclude that for $n \geq n(r)$, we have $\triangle^r\overline{p}(n) > 0$. This completes the proof. □

Up to now, we have shown the positivity of the $\triangle^r\overline{p}(n)$. In fact, using the inequality (2.11), we can also give the following upper bound for $\triangle^r\overline{p}(n)$.

Theorem 2.3. *For $r \geq 1$,*

$$\triangle^r\overline{p}(n) \leq 2^{r-3}\left(1-2^{-\frac{3}{2}}\right)\zeta(3/2)\frac{e^{\pi\sqrt{n+r}}}{n+r}.$$

Proof. Recall that inequality (2.11) states that for $r \geq 1$ and $k \geq 1$,

$$|\triangle^r f_k(n)| \leq \frac{1}{16}\pi^{\frac{5}{2}}2^r\cdot k^{-\frac{3}{2}}L_{3/2}\left(\frac{\pi^2}{4}(n+r)\right).$$

Thus, we find that for $r \geq 1$,

$$\begin{aligned}\triangle^r \overline{p}(n) &\leq \sum_{k\geq 1, 2\nmid k} |\triangle^r f_k(n)| \\ &\leq \frac{1}{16}\pi^{\frac{5}{2}} 2^r L_{3/2}\left(\frac{\pi^2}{4}(n+r)\right)\left(\sum_{k\geq 1} k^{-\frac{3}{2}} - \sum_{k\geq 1, 2|k} k^{-\frac{3}{2}}\right) \\ &\leq \frac{1}{16}\pi^{\frac{5}{2}} 2^r L_{3/2}\left(\frac{\pi^2}{4}(n+r)\right)\left(\zeta(3/2) - 2^{-\frac{3}{2}}\zeta(3/2)\right) \\ &\leq \frac{1}{16}\pi^{\frac{5}{2}} 2^r \left(1 - 2^{-\frac{3}{2}}\right)\zeta(3/2) L_{3/2}\left(\frac{\pi^2}{4}(n+r)\right).\end{aligned}$$

Using the following inequality due to Almkvist [3]

$$L_{3/2}(x) \leq \frac{1}{2\sqrt{\pi}} \frac{e^{2\sqrt{x}}}{x},$$

we obtain that

$$\begin{aligned}\triangle^r \overline{p}(n) &\leq \frac{1}{16}\pi^{\frac{5}{2}} 2^r \left(1 - 2^{-\frac{3}{2}}\right)\zeta(3/2) \frac{1}{2\sqrt{\pi}} \frac{e^{2\sqrt{\pi^2(n+r)/4}}}{\pi^2(n+r)/4} \\ &\leq 2^{r-3}\left(1 - 2^{-\frac{3}{2}}\right)\zeta(3/2)\frac{e^{\pi\sqrt{n+r}}}{n+r}.\end{aligned}$$

This completes the proof. □

Note that $\triangle^r \overline{p}(n)$ really grow exponentially. Hence, as a conclusion of this section, we propose the following open problem.

Problem 2.4. *Find a sharp lower bound for* $\triangle^r \overline{p}(n)$.

3. The positivity of $(-1)^{r-1}\triangle^r \log \overline{p}(n)$

In this section, we shall prove that for any given $r \geq 1$, there exists a positive number $n(r)$ such that for $n > n(r)$, $(-1)^{r-1}\triangle^r \log \overline{p}(n)$ is positive.

Theorem 3.1. *For each* $r \geq 1$, *there exists a positive integer* $n(r)$ *such that for* $n \geq n(r)$,

$$(-1)^{r-1}\triangle^r \log \overline{p}(n) > 0.$$

Proof. The case $r = 1$ is trivial since $\overline{p}(n+1) > \overline{p}(n)$ for $n \geq 1$. For $r = 2$, Engel [11] has shown that $\overline{p}(n)$ is log-concave for $n \geq 2$, namely, for $n \geq 2$,

$$-\triangle^2 \log \overline{p}(n) \geq 0.$$

We now consider the case $r \geq 3$. Notice that

$$\begin{aligned}\frac{d}{dn}\left(\frac{\sinh(\hat{\mu}(n)/k)}{\sqrt{n}}\right) &= \frac{\pi}{2kn}\left(\cosh\left(\frac{\hat{\mu}(n)}{k}\right) - \frac{k}{\hat{\mu}(n)}\sinh\left(\frac{\hat{\mu}(n)}{k}\right)\right)\\ &= \frac{\pi}{4kn}\left(\left(1+\frac{k}{\hat{\mu}(n)}\right)e^{\frac{-\hat{\mu}(n)}{k}} + \left(1-\frac{k}{\hat{\mu}(n)}\right)e^{\frac{\hat{\mu}(n)}{k}}\right),\end{aligned}$$

where $\hat{\mu}(n) = \pi\sqrt{n}$, we can rewrite (1.9) as

$$\overline{p}(n) = \frac{1}{8n}\sum_{\substack{k\geq 1\\ 2\nmid k}} \frac{1}{\sqrt{k}}\hat{A}_k(n)\left(\left(1+\frac{k}{\hat{\mu}(n)}\right)e^{\frac{-\hat{\mu}(n)}{k}} + \left(1-\frac{k}{\hat{\mu}(n)}\right)e^{\frac{\hat{\mu}(n)}{k}}\right). \tag{3.1}$$

Recall that $\hat{A}_1(n) = 1$ in (1.12), we split $\overline{p}(n)$ into two terms as Engel [11]

$$\overline{p}(n) = \hat{T}(n) + \hat{R}(n), \tag{3.2}$$

where

$$\hat{T}(n) = \frac{1}{8n}\left(e^{-\hat{\mu}(n)} + \left(1-\frac{1}{\hat{\mu}(n)}\right)e^{\hat{\mu}(n)}\right), \tag{3.3}$$

$$\hat{R}(n) = \frac{e^{-\hat{\mu}(n)}}{8n\hat{\mu}(n)} + R_2(n,2). \tag{3.4}$$

Restate (3.2) as

$$\overline{p}(n) = \hat{T}(n)\left(1+\frac{\hat{R}(n)}{\hat{T}(n)}\right). \tag{3.5}$$

Applying (3.3) to (3.5) and taking the logarithm of both sides, we have that

$$\begin{aligned}\log \overline{p}(n) = \log\frac{\pi^2}{8} &- 3\log\hat{\mu}(n) + \log(\hat{\mu}(n)-1) + \hat{\mu}(n)\\ &+ \log\left(1+\frac{\hat{\mu}(n)}{\hat{\mu}(n)-1}e^{-2\hat{\mu}(n)}\right) + \log\left(1+\frac{\hat{R}(n)}{\hat{T}(n)}\right).\end{aligned}$$

Hence, $(-1)^{r-1}\triangle^r \log \overline{p}(n)$ can be expressed as

$$(-1)^{r-1}\triangle^r \log \overline{p}(n) = H_r + F_1 + F_2, \tag{3.6}$$

where

$$H_r = (-1)^{r-1}\triangle^r(-3\log\hat{\mu}(n) + \log(\hat{\mu}(n)-1) + \hat{\mu}(n)), \tag{3.7}$$

$$F_1 = (-1)^{r-1}\triangle^r \log\left(1 + \frac{\hat{\mu}(n)}{\hat{\mu}(n)-1}e^{-2\hat{\mu}(n)}\right), \tag{3.8}$$

$$F_2 = (-1)^{r-1}\triangle^r \log\left(1 + \frac{\hat{R}(n)}{\hat{T}(n)}\right). \tag{3.9}$$

Let

$$G_r = F_1 + F_2, \tag{3.10}$$

then we have that for $r \geq 1$,

$$(-1)^{r-1}\triangle^r \log\overline{p}(n) \geq H_r - |G_r|. \tag{3.11}$$

To estimate the lower bound for $(-1)^{r-1}\triangle^r \log\overline{p}(n)$, we shall give a lower bound for H_r and an upper bound for $|G_r|$. We first concern with $|G_r|$ and get the following upper bound for $|G_r|$.

Lemma 3.2. *For* $n \geq 225$, *we have*

$$|G_r| \leq 2^{r+1}e^{-\frac{\hat{\mu}(n)}{2}}. \tag{3.12}$$

To prove Lemma 3.2, we need to give upper bounds for $|F_1|$ and $|F_2|$. Recall that for any function $f(n)$,

$$\triangle^r f(n) = \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}f(n+k),$$

we have that

$$F_1 = \sum_{k=0}^{r}(-1)^{k+1}\binom{r}{k}\log\left(1 + \frac{\hat{\mu}(n+k)}{\hat{\mu}(n+k)-1}e^{-2\hat{\mu}(n+k)}\right).$$

So,

$$|F_1| \leq \sum_{k=0}^{r}\binom{r}{k}\log\left(1 + \frac{\hat{\mu}(n+k)}{\hat{\mu}(n+k)-1}e^{-2\hat{\mu}(n+k)}\right). \tag{3.13}$$

It is easily seen that $1 + \frac{\hat{\mu}(n)}{\hat{\mu}(n)-1}e^{-2\hat{\mu}(n)}$ decreases with n for $n \geq 1$. Thus, we have that for $n \geq 1$ and $0 \leq k \leq r$,

$$\log\left(1 + \frac{\hat{\mu}(n+k)}{\hat{\mu}(n+k)-1}e^{-2\hat{\mu}(n+k)}\right) \leq \log\left(1 + \frac{\hat{\mu}(n)}{\hat{\mu}(n)-1}e^{-2\hat{\mu}(n)}\right). \tag{3.14}$$

Applying (3.14) to (3.13), we obtain that for $n \geq 1$,

$$|F_1| \leq 2^r \log\left(1 + \frac{\hat{\mu}(n)}{\hat{\mu}(n) - 1} e^{-2\hat{\mu}(n)}\right). \tag{3.15}$$

It is easily verified that for $x \geq 0$,

$$\log(1 + x) \leq x. \tag{3.16}$$

So we have that for $n \geq 1$,

$$|F_1| \leq 2^r \frac{\hat{\mu}(n)}{\hat{\mu}(n) - 1} e^{-2\hat{\mu}(n)}. \tag{3.17}$$

Now we turn to $|F_2|$. Let us first find appropriate bounds for $\hat{R}(n)$ and $\hat{T}(n)$, which will be used in the estimation of $|F_2|$. By (1.11) and (3.4), we have

$$\begin{aligned} |\hat{R}(n)| &\leq \left|\frac{e^{-\hat{\mu}(n)}}{8n\hat{\mu}(n)}\right| + |R_2(n, 2)| \\ &\leq \frac{e^{-\hat{\mu}(n)}}{8n\hat{\mu}(n)} + \frac{2^{5/2}}{n\hat{\mu}(n)} \sinh\left(\frac{\hat{\mu}(n)}{2}\right) \\ &\leq \frac{\left(\frac{e^{-\frac{\hat{\mu}(n)}{2}}}{8} - 1\right) e^{-\frac{\hat{\mu}(n)}{2}} + 2^{3/2} e^{\frac{\hat{\mu}(n)}{2}}}{n\hat{\mu}(n)} \\ &\leq \frac{2^{3/2}}{n\hat{\mu}(n)} e^{\frac{\hat{\mu}(n)}{2}}. \end{aligned} \tag{3.18}$$

Recall that

$$\hat{T}(n) = \frac{1}{8n}\left(e^{-\hat{\mu}(n)} + \left(1 - \frac{1}{\hat{\mu}(n)}\right) e^{\hat{\mu}(n)}\right).$$

Since $\hat{\mu}(n) > \pi$, we have that

$$\begin{aligned} \hat{T}(n) &> \frac{1}{8n}\left(1 - \frac{1}{\hat{\mu}(n)}\right) e^{\hat{\mu}(n)} \\ &> \frac{1}{8n}\left(1 - \frac{1}{\pi}\right) e^{\hat{\mu}(n)} \\ &> \frac{1}{16n} e^{\hat{\mu}(n)} > 1. \end{aligned} \tag{3.19}$$

Thus, by (3.18) and (3.19), we see that for $n \geq 3$,

$$0 < \frac{|\hat{R}(n)|}{\hat{T}(n)} \leq \frac{2^{11/2}}{\hat{\mu}} e^{-\frac{\hat{\mu}}{2}} < 1. \tag{3.20}$$

Now we proceed to estimate $|F_2|$. By (3.9) and (2.10), we have

$$|F_2| \leq \sum_{k=0}^{r} \binom{r}{k} \left| \log\left(1 + \frac{\hat{R}(n+k)}{\hat{T}(n+k)}\right) \right|. \tag{3.21}$$

For $0 < x < 1$, it can be easily checked that

$$|\log(1 \pm x)| \leq -\log(1-x). \tag{3.22}$$

Then we deduce that for $n \geq 3$,

$$\left| \log\left(1 + \frac{\hat{R}(n+k)}{\hat{T}(n+k)}\right) \right| \leq -\log\left(1 - \frac{|\hat{R}(n+k)|}{\hat{T}(n+k)}\right). \tag{3.23}$$

Thus, for $n \geq 3$,

$$|F_2| \leq -\sum_{k=0}^{r} \binom{r}{k} \log\left(1 - \frac{|\hat{R}(n+k)|}{\hat{T}(n+k)}\right). \tag{3.24}$$

Since $-\log(1-x)$ is increasing for $x > -1$, combining (3.20) and (3.24), we get that for $n \geq 3$,

$$|F_2| \leq -\sum_{k=0}^{r} \binom{r}{k} \log\left(1 - \frac{2^{11/2}}{\hat{\mu}(n+k)} e^{-\frac{\hat{\mu}(n+k)}{2}}\right). \tag{3.25}$$

It can be checked that $\frac{2^{11/2}}{\hat{\mu}(n+k)} e^{-\frac{\hat{\mu}(n+k)}{2}}$ decreases with n. Thus we have that for $n \geq 3$,

$$|F_2| \leq -2^r \log\left(1 - \frac{2^{11/2}}{\hat{\mu}(n)} e^{-\frac{\hat{\mu}(n)}{2}}\right). \tag{3.26}$$

In view of the fact that for $0 < x < 1$,

$$\log(1-x) \geq \frac{-x}{1-x}, \tag{3.27}$$

from (3.26), we get that for $n \geq 3$,

$$|F_2| \leq 2^r \left(\frac{\frac{2^{11/2}}{\hat{\mu}(n)} e^{-\frac{\hat{\mu}(n)}{2}}}{1 - \frac{2^{11/2}}{\hat{\mu}(n)} e^{-\frac{\hat{\mu}(n)}{2}}} \right) = 2^r e^{-\frac{\hat{\mu}(n)}{2}} \left(\frac{2^{11/2}}{\hat{\mu}(n) - 2^{11/2} e^{-\frac{\hat{\mu}(n)}{2}}} \right). \tag{3.28}$$

Since $\hat{\mu}(n)$ increases with n and $e^{-\frac{\hat{\mu}(n)}{2}}$ decreases with n, it can be checked that for $n \geq 225$,

$$0 < \frac{2^{11/2}}{\hat{\mu}(n) - 2^{11/2}e^{-\frac{\hat{\mu}(n)}{2}}} < 1. \tag{3.29}$$

So we have that for $n \geq 225$,

$$|F_2| \leq 2^r e^{-\frac{\hat{\mu}(n)}{2}}. \tag{3.30}$$

Combining (3.17) and (3.30), we obtain that for $n \geq 225$,

$$\begin{aligned} |G_r| &\leq |F_1| + |F_2| \\ &\leq 2^r \frac{\hat{\mu}(n)}{\hat{\mu}(n) - 1} e^{-2\hat{\mu}(n)} + 2^r e^{-\frac{\hat{\mu}(n)}{2}} \\ &\leq 2^r e^{-\frac{\hat{\mu}(n)}{2}} \left(\frac{\hat{\mu}(n)}{\hat{\mu}(n) - 1} e^{-\frac{3\hat{\mu}(n)}{2}} + 1 \right) \\ &\leq 2^{r+1} e^{-\frac{\hat{\mu}(n)}{2}}. \end{aligned} \tag{3.31}$$

This completes the proof. □

To estimate H_r, we introduce the following proposition due to Odlyzko [22].

Proposition 3.3. *Let r be a positive integer. Suppose that $f(x)$ is a function with infinite continuous derivatives for $x \geq 1$, and $(-1)^{k-1} f^{(k)}(x) > 0$ for $k \geq 1$. Then for $r \geq 1$,*

$$(-1)^{r-1} f^{(r)}(x + r) \leq (-1)^{r-1} \triangle^r f(x) \leq (-1)^{r-1} f^{(r)}(x). \tag{3.32}$$

We are ready to estimate H_r. Since for $n \geq 1$,

$$\log(\hat{\mu}(n) - 1) - \log \hat{\mu}(n) = -\sum_{k=1}^{\infty} \frac{1}{k\hat{\mu}(n)^k},$$

equality (3.7) can be rewritten as

$$H_r = (-1)^{r-1} \triangle^r (\hat{\mu}(n) - 2\log \hat{\mu}(n)) - \sum_{k=1}^{\infty} (-1)^{r-1} \triangle^r \left(\frac{1}{k\hat{\mu}(n)^k} \right). \tag{3.33}$$

It is easily seen that the rth derivatives of $\hat{\mu}(x) = \pi\sqrt{x}$, $\log \hat{\mu}(x)$ and $\hat{\mu}(x)^{-k}$ are given as follows

$$\hat{\mu}^{(r)}(x) = \frac{\pi}{2}(-1)^{r-1} \left(\frac{1}{2}\right)_{r-1} \frac{1}{x^{r-\frac{1}{2}}}, \tag{3.34}$$

$$\log^{(r)} \hat{\mu}(x) = \frac{1}{2}(-1)^{r-1} \frac{(r-1)!}{x^r}, \tag{3.35}$$

$$\left(\frac{1}{\hat{\mu}(x)^k}\right)^{(r)} = \frac{1}{\pi^k}(-1)^r \left(\frac{k}{2}\right)_r \frac{1}{x^{r+\frac{k}{2}}}. \tag{3.36}$$

Hence, by Proposition 3.3, we find that for $r \geq 1$ and $k \geq 1$,

$$\begin{aligned} H_r &\geq \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1} \frac{1}{(n+r)^{r-\frac{1}{2}}} - \frac{(r-1)!}{n^r} + \sum_{k=1}^{\infty} \frac{1}{k\pi^k}\left(\frac{k}{2}\right)_r \frac{1}{(n+r)^{r+\frac{k}{2}}} \\ &\geq \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1} \frac{1}{(n+r)^{r-\frac{1}{2}}} - \frac{(r-1)!}{n^r}. \end{aligned} \tag{3.37}$$

Observe that for $n \geq r$,

$$\frac{1}{(n+r)^{r-\frac{1}{2}}} \geq \frac{1}{(2n)^{r-\frac{1}{2}}}.$$

It follows that for $n \geq r$,

$$H_r \geq \frac{a_1}{n^{r-\frac{1}{2}}} - \frac{(r-1)!}{n^r}, \tag{3.38}$$

where

$$a_1 = \frac{\pi}{2^{r+\frac{1}{2}}}\left(\frac{1}{2}\right)_{r-1}.$$

Up to now, we have given an upper bound for $|G_r|$ and a lower bound for H_r. We proceed to focus on $(-1)^{r-1}\triangle^r \log \overline{p}(n)$. Applying (3.12) and (3.38) to (3.11) yields that for $n \geq \max\{225, r\}$,

$$(-1)^{r-1}\triangle^r \log \overline{p}(n) \geq H_r - |G_r| \geq \frac{a_1}{n^{r-\frac{1}{2}}} - \frac{(r-1)!}{n^r} - 2^{r+1}e^{-\frac{\hat{\mu}(n)}{2}}.$$

It can be seen that for $n \geq \frac{4((r+1)!)^2}{a_1^2} + 1$,

$$\frac{(r-1)!}{n^r} < \frac{a_1}{2n^{r-\frac{1}{2}}}.$$

Thus, we get that for $n \geq \max\left\{225, r, \frac{4((r+1)!)^2}{a_1^2} + 1\right\}$,

$$(-1)^{r-1}\triangle^r \log \overline{p}(n) > \frac{a_1}{2n^{r-\frac{1}{2}}} - 2^{r+1}e^{-\frac{\hat{\mu}(n)}{2}}. \tag{3.39}$$

To prove the positivity of $(-1)^{r-1}\triangle^r \log \overline{p}(n)$, we consider the following equation

$$\frac{a_1}{2x^{r-\frac{1}{2}}} = 2^{r+1}e^{-\frac{\hat{\mu}(x)}{2}}. \tag{3.40}$$

We claim the equation has two real roots

$$x_1 = \frac{4(2r-1)^2}{\pi^2}\left(W_0\left(-\frac{\pi}{4(2r-1)}\left(\frac{\pi}{8\sqrt{2}}\left(\frac{1}{2}\right)_{r-1}\right)^{\frac{1}{2r-1}}\right)\right)^2, \tag{3.41}$$

$$x_2 = \frac{4(2r-1)^2}{\pi^2}\left(W_{-1}\left(-\frac{\pi}{4(2r-1)}\left(\frac{\pi}{8\sqrt{2}}\left(\frac{1}{2}\right)_{r-1}\right)^{\frac{1}{2r-1}}\right)\right)^2, \tag{3.42}$$

where $W_0(z)$ and $W_{-1}(z)$ are two branches of Lambert W function $W(z)$, see Corless, Gonnet, Hare, Jeffrey and Knuth [7].

By the property of Lambert W function, we know that for any Lambert W function $W(z)$, $W(z)$ has two real values $W_0(z)$ and $W_{-1}(z)$ for $-\frac{1}{e} < z < 0$. Using the following inequality given by Robbins [24]

$$r! < \sqrt{2\pi} r^{r+\frac{1}{2}} e^{-r+\frac{1}{12r}}, \tag{3.43}$$

we obtain that

$$-\frac{1}{e} < -\frac{\pi}{4(2r-1)}\left(\frac{\pi}{8\sqrt{2}}\left(\frac{1}{2}\right)_{r-1}\right)^{\frac{1}{2r-1}} < 0. \tag{3.44}$$

So equation (3.40) has two real roots. Let $b = \max\{x_1, x_2\}$ be the larger real root. It follows that for $n \geq b + 1$,

$$\frac{a_1}{2n^{r-\frac{1}{2}}} - 2^{r+1} e^{-\frac{\hat{\mu}(n)}{2}} > 0. \tag{3.45}$$

Let

$$n(r) = \max\left\{225, r, \frac{4((r+1)!)^2}{a_1^2} + 1, b + 1\right\}.$$

Combining (3.39) and (3.45), we conclude that for $n \geq n(r)$,

$$(-1)^{r-1} \triangle^r \log \overline{p}(n) > 0.$$

This completes the proof. □

Note that Theorem 3.1 means that for any r, there exists $n'(r)$ such that for $n > n'(r)$ we have $\frac{1}{\overline{p}(n)}$ is log-monotonic of order r. (For more background for log-monotonic sequences, see [5].) Furthermore, we also wish to seek for a sharp lower bound for $(-1)^{r-1} \triangle^r \log \overline{p}(n)$.

Problem 3.4. *If there exists a positive number A such that*

$$\frac{(-1)^{r-1} \triangle^r \log \overline{p}(n)}{n^{-\frac{r-1}{2}}} > A,$$

for any r and sufficiently large n?

4. An upper bound for $(-1)^{r-1}\triangle^r \log \overline{p}(n)$

In this section, we give an upper bound for $(-1)^{r-1}\triangle^r \log \overline{p}(n)$.

Theorem 4.1. *For each $r \geq 1$, there exists a positive integer $n(r)$ such that for $n \geq n(r)$,*

$$(-1)^{r-1}\triangle^r \log \overline{p}(n) < \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1} \frac{1}{n^{r-\frac{1}{2}}}. \tag{4.1}$$

Proof. First, we treat the case $r = 1$, which states that for $n \geq 36$

$$\triangle \log \overline{p}(n) < \frac{\pi}{2\sqrt{n}}.$$

Since the upper bound for $|G_r|$ has been given in (3.12). We only need to find an appropriate upper bound for H_r for $r = 1$. By Proposition 3.3, we have

$$H_1 \leq \frac{\pi}{2\sqrt{n}} - \frac{3}{2(n+1)} + \frac{\pi}{2\sqrt{n}(\pi\sqrt{n}-1)}. \tag{4.2}$$

Combining (4.2) and the upper bound for $|G_1|$ in (3.12) leads to that for $n \geq 1$,

$$\triangle \log \overline{p}(n) \leq H_1 + |G_1| \leq \frac{\pi}{2\sqrt{n}} - \frac{3}{2(n+1)} + \frac{\pi}{2\sqrt{n}(\pi\sqrt{n}-1)} + 4e^{-\frac{\hat{\mu}(n)}{2}}. \tag{4.3}$$

We proceed to estimate the last three terms of the right hand side of (4.3). For the second term, it is easily seen that for $n \geq 1$,

$$-\frac{3}{2(n+1)} \leq -\frac{3}{4n}. \tag{4.4}$$

For the third term of the right hand side of (4.3). Since $\pi\sqrt{n} > 4$ for $n \geq 2$, we have that for $n \geq 2$,

$$\frac{\pi}{2\sqrt{n}(\pi\sqrt{n}-1)} = \frac{1}{2n(1-\frac{1}{\pi\sqrt{n}})} < \frac{2}{3n}. \tag{4.5}$$

For the last term of the right hand side of (4.3). Notice that for $n \geq 36$,

$$4e^{-\frac{\hat{\mu}(n)}{2}} < \frac{1}{12n}. \tag{4.6}$$

Combining (4.3)–(4.6), we see that for $n \geq 36$,

$$\triangle \log \overline{p}(n) < \frac{\pi}{2\sqrt{n}}.$$

We now turn to the case $r \geq 2$. Recall that

$$(-1)^{r-1}\triangle^r \log \overline{p}(n) = H_r + G_r,$$

thus for $n \geq 1$,

$$(-1)^{r-1}\triangle^r \log \overline{p}(n) \leq H_r + |G_r|. \tag{4.7}$$

Now we need an upper bound for H_r. Applying Proposition 3.3 to (3.33), we get

$$H_r \leq \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1}\frac{1}{n^{r-\frac{1}{2}}} - \frac{(r-1)!}{(n+r)^r} + \sum_{k=1}^{\infty}\frac{1}{k\pi^k}\left(\frac{k}{2}\right)_r \frac{1}{n^{r+\frac{k}{2}}}. \tag{4.8}$$

Notice that for $n \geq r$,

$$\frac{1}{(n+r)^r} \geq \frac{1}{(2n)^r},$$
$$\frac{1}{n^{r+\frac{k}{2}}} \leq \frac{1}{n^{r+\frac{1}{2}}} \cdot \frac{1}{r^{\frac{k}{2}-\frac{1}{2}}},$$

we have that for $n \geq r$,

$$H_r \leq \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1}\frac{1}{n^{r-\frac{1}{2}}} - \frac{(r-1)!}{2^r n^r} + \frac{a_2}{n^{r+\frac{1}{2}}}, \tag{4.9}$$

where

$$a_2 = \sum_{k=1}^{\infty}\frac{1}{k\pi^k}\left(\frac{k}{2}\right)_r \frac{1}{r^{\frac{k}{2}-\frac{1}{2}}}.$$

Obviously, a_2 is convergent and hence a finite number. It can be checked that for $n \geq \frac{4a_2^2 4^r}{((r-1)!)^2} + 1$,

$$\frac{a_2}{n^{r+\frac{1}{2}}} < \frac{(r-1)!}{2^{r+1}n^r}. \tag{4.10}$$

Combining (4.9) and (4.10) yields that for $n \geq \max\left\{r, \frac{4a_2^2 4^r}{((r-1)!)^2} + 1\right\}$,

$$H_r \leq \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1}\frac{1}{n^{r-\frac{1}{2}}} - \frac{(r-1)!}{2^{r+1}n^r}.$$

So we have that for $n \geq \max\left\{225, r, \frac{4a_2^2 4^r}{((r-1)!)^2} + 1\right\}$,

$$(-1)^{r-1}\triangle^r \log \overline{p}(n) \leq H_r + |G_r| \\ \leq \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1}\frac{1}{n^{r-\frac{1}{2}}} - \frac{(r-1)!}{2^{r+1}n^r} + 2^{r+1}e^{\frac{-\hat{\mu}(n)}{2}}. \tag{4.11}$$

It can be checked that for sufficiently large n,

$$\frac{(r-1)!}{2^{r+1}n^r} > 2^{r+1}e^{\frac{-\hat{\mu}(n)}{2}}.$$

Thus, from (4.11), we assert that for sufficiently large n,

$$(-1)^{r-1}\triangle^r \log \overline{p}(n) < \frac{\pi}{2}\left(\frac{1}{2}\right)_{r-1}\frac{1}{n^{r-\frac{1}{2}}}.$$

In order to estimate $n(r)$, we proceed to consider the following equation

$$\frac{(r-1)!}{2^{r+1}x^r} = 2^{r+1}e^{\frac{-\hat{\mu}(x)}{2}}. \tag{4.12}$$

Similar to (3.40), the solution of (4.12) has the following form

$$x = \frac{16r^2}{\pi^2}\left(W\left(-\frac{\pi}{8r}\left(\frac{(r-1)!}{4}\right)^{\frac{1}{2r}}\right)\right)^2. \tag{4.13}$$

By (3.43), we obtain that for $r \geq 2$,

$$-\frac{1}{e} < -\frac{\pi}{8r}\left(\frac{(r-1)!}{4}\right)^{\frac{1}{2r}} < 0. \tag{4.14}$$

Thus, (4.12) has two real roots. Let x_1 be the larger real root. Thus, for $n \geq x_1 + 1$,

$$\frac{(r-1)!}{2^{r+1}n^r} > 2^{r+1}e^{\frac{-\hat{\mu}(n)}{2}}. \tag{4.15}$$

Combining (4.15) and (4.11), we conclude that (4.1) holds for $n \geq n(r)$, where

$$n(r) = \max\left\{225, r, \frac{4a_2^2 4^r}{((r-1)!)^2} + 1, x_1 + 1\right\}.$$

This completes the proof of Theorem 4.1. □

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