Context-free Grammars for Combinatorial Enumeration

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Outline

I. What is a Context-Free Grammar?

What is a context-free grammar?

The grammars we will be talking about are exactly those as in computer science, that is, a set of substitution rules over an alphabet. For the purpose of combinatorial enumeration, we treat the alphabet as a set of commuting variables. The key idea is that the notion of a context-free grammar is closely connected to the notion of a derivative for which the Leibniz formula holds. The production rules are supposed to be applied to Laurent polynomials over a set of variables.

Let us give some simple examples to illustrate the idea. Let $V = \{a, x\}$ be a set of variable, and let

$$G = \{a \to ax, \quad x \to x\}.$$

be a context-free grammar, and let D be the formal derivative with respect to the grammar G.

A formal derivative can be formulated as a differential operator

$$D = ax\frac{\partial}{\partial a} + x\frac{\partial}{\partial x}.$$

The Leibniz formula:

$$D^n(fg) = \sum_{k=0}^{n} \binom{n}{k} D^k(f) D^{n-k}(g).$$

It should be noted that the above expression of a formal derivative in terms of a differential operator can be employed in the implementation in Maple for the computation of the action of D on a Laurent polynomial $f(a, x)$.

A context-free grammar has two facets. One is combinatorial, and the other is computational.

What is a context-free grammar?

For the above grammar $G = \{a \to ax, \quad x \to x\}$, in combinatorial terms one can easily argue that

$$D^n(a) = a \sum_{k=1}^{n} S(n,k) x^k,$$

where $S(n,k)$ are the Stirling numbers of the second kind.

$$\begin{aligned} D(a) &= ax, \\ D^2(a) &= D(ax) = D(a)x + aD(x) = ax + ax^2, \\ D^3(a) &= D(ax) + D(a)x^2 + aD(x^2), \\ D(x^2) &= 2xD(x) = 2x^2. \end{aligned}$$

Thus,

$$D^3(a) = ax + ax^2 + a2x^2 + ax^3 = ax + 3ax^2 + ax^3.$$

Remarks

Remark 1. One feature of a context-free grammar is that it makes use of many variables, like x and y in this case, but we use only one derivative D involving many variables and we consider the generating function only in one variable. It turns out that many variables may work better than only one variable. In some sense, these variables may be viewed as parameters.

Remark 2. The variables of a grammar can be associated as tags or marks with a combinatorial structure. As a result, a grammar can be employed to study statistics of combinatorial objects such as permutations, partitions and trees.

Since D is a derivative, by the Leibniz formula and the relation

$$D^{m+n} = D^m D^n,$$

we obtain the convolution formula for the Stirling numbers of the second kind, due to Verde-Star (1988), see Chen (1993): For $m, n \geq 1$,

$$S(m+n,k) = \sum_{i+j\geq k} \binom{m}{j} i^{m-j} S(n,i) S(j,k-i).$$

L. Verde-Star, Interpolation and combinatorial functions, Stud. Appl. Math., 79 (1988), 65-92.

W.Y.C. Chen, Context-free grammars, differential operators and formal power series, Theoret. Comput. Sci. 117 (1993), 113-129.

Spivey's Bell number formula

This relation implies the following identity on Bell numbers, which has been called Spivey's Bell number formula (Spivey, 2008).

$$B_{n+m} = \sum_{j=0}^{m} \sum_{k=0}^{n} k^{m-j} S(n,k) \binom{m}{j} B_j.$$

I came across to the work of Verde-Star, largely because I met him at MIT during his visit to Gian-Carlo Rota, while I was a student.

M. Spivey, A generalized recurrence for Bell numbers, J. Integer Seq., 11 (2008), Article 08.2.5.

Connection to the symbolic method

The key to the symbolic method is to treat a sequence $a_0, a_1, a_2, \ldots$ like $a^0, a^1, a^2, \ldots$. Rota made it rigorous in light of the idea of linear functionals.

Grammars can be used to put the classic argument (or the umbral calculus for the Bell polynomials) on a firm foundation, see Johnson:

W.P. Johnson, The curious history of Faà di Bruno's formula, Amer. Math. Monthly 109 (2002) 217-234.

Connection to the symbolic method

Let G be the grammar

$$G = \{f \to f, a_i \to a_{i+1} \,|\, i = 0, 1, 2, \ldots\}.$$

Let D denote the derivative with respect to the grammar G. Bear in mind that the a_i are considered as variables. Clearly, we have

$$D(f^{-1}) = -f^{-1}.$$

Assume that

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k,$$

for $n \geq 0$. Then the above relation can be recast as

$$b_n = f^{-1} D^n (f a_0).$$

Consequently,

$$a_n = D^n(a_0) = D^n(f^{-1} f a_0),$$

which equals

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f^{-1} D^k (f a_0) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k.$$

II. The Dumont Grammar for the Eulerian Numbers

The Eulerian numbers

Definition

For $1 \leq k \leq n$, the Eulerian numbers $A(n,k)$ are defined to be the number of permutations of $[n] = \{1, 2, \ldots, n\}$ with k descents, where we assume that the last position is always a descent. For $n = 0$, we define $A(0,0) = 1$.

The Eulerian numbers are often denoted by $\left\langle {n \atop k} \right\rangle$. For $n \geq 0$, the Eulerian polynomials $A_n(x)$ are defined by $A_0(x) = 1$ and for $n \geq 1$,

$$A_n(x) = \sum_{\sigma \in S_n} x^{\operatorname{des}(\sigma)} = \sum_{k=1}^{n} A(n,k) x^k,$$

where $\operatorname{des}(\sigma)$ denotes the number of descents of a permutation σ.

Dumont's grammar

Dumont's Grammar. Dumont discovered the following grammar for the Eulerian polynomials $A_n(x)$:

$$G = \{x \to xy, \quad y \to xy\}.$$

Let D denote the derivative with respect to the above grammar G. If we do not mind the rigor in notation, for $n \geq 0$, we use $A_n(x, y)$ to the denote the bivariate polynomial $D^n(y)$, namely,

$$A_n(x, y) = D^n(y).$$

Example

The first few values of $A_n(x, y)$ are given below

$$
\begin{aligned}
A_0(x,y) &= y,\\
A_1(x,y) &= xy,\\
A_2(x,y) &= xy^2 + x^2y,\\
A_3(x,y) &= xy^3 + 4x^2y^2 + x^3y,\\
A_4(x,y) &= xy^4 + 11x^2y^3 + 11x^3y^2 + x^4y,\\
A_5(x,y) &= xy^5 + 26x^2y^4 + 66x^3y^3 + 26x^4y^2 + x^5y,\\
A_6(x,y) &= xy^6 + 57x^2y^5 + 302x^3y^4 + 302x^4y^3 + 57x^5y^2 + x^6y.
\end{aligned}
$$

The Dumont grammar for the Eulerian numbers

Theorem (Dumont)

For $n \geq 1$,

$$A_n(x, y) = \sum_{\sigma \in S_n} x^{\mathrm{des}\,(\sigma)} y^{\mathrm{asc}\,(\sigma)},$$

where $\mathrm{asc}\,(\sigma)$ *stands for the number of ascents of* σ *and we always assume that the first position is an ascent. In other words,* i *is an ascent of* $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ *if* $i = 1$ *or* $\sigma_{i-1} < \sigma_i$.

Since $A_0(x, y) = y$, the above theorem implies that for $n \geq 0$,

$$A_n(x) = A_n(x, y)|_{y=1}.$$

It is not difficult to see that $A_n(x, y)$ can also be expressed in terms of permutations in S_n. For a permutation π in S_n, we give a labeling of π as follows. For $0 \leq i \leq n$, if $\pi_i < \pi_{i+1}$, we label π_i by x; if $\pi_i > \pi_{i+1}$, we label π_i by y. The weight of π is defined as the product of the labels, that is,

$$w(\pi) = x^{\mathrm{asc}(\pi)} y^{\mathrm{des}(\pi)},$$

where $\mathrm{asc}(\pi)$ denotes the number of ascents in π and $\mathrm{des}(\pi)$ denotes the number of descents in π. For $n \geq 1$, it can be shown that the polynomial $A_n(x, y)$ permits the following equivalent expression:

$$A_n(x, y) = \sum_{\pi \in S_n} x^{\mathrm{asc}(\pi)} y^{\mathrm{des}(\pi)}.$$

Example of a grammatical labeling

To illustrate the relation between the action of the formal derivative D and the insertion of the element $n+1$ into a permutation on $[n]$, we give the following example. Let $n=6$ and $\pi = 325641$. The grammatical labeling of π reads

$$\begin{array}{ccccccccccccc} & 3 & & 2 & & 5 & & 6 & & 4 & & 1 & \\ x & & y & & x & & x & & y & & y & & y, \end{array}$$

where the element 0 is made invisible. If we insert 7 after 5, the resulting permutation and its grammatical labeling are given below,

$$\begin{array}{ccccccccccccccc} & 3 & & 2 & & 5 & & 7 & & 6 & & 4 & & 1 & \\ x & & y & & x & & x & & y & & y & & y & & y. \end{array}$$

Example of a grammatical labeling

Notice that the insertion of 7 after 5 corresponds to applying the rule $x \to xy$ to the label x associated with 5. We have a similar situation when the new element is inserted after an element labeled by y. Hence the action of the formal derivative D on the set of weights of permutations in S_n gives the set of weights of permutations in S_{n+1}. This yields a grammatical expression for $A_n(x, y)$.

D. Dumont

Dumont made ingenious discoveries of grammars in connection with classical problems on permutations and trees. These grammars take astonishingly simple forms. As far as enumeration is concerned, a grammar furnishes sufficient information. In other words, Dumont found remarkable generation rules for several combinatorial objects.

D. Dumont

I was thrilled by his results. Out of great admiration, I was tempted to invite him to visit China. He kindly accepted the invitation in 2014, but it was sad that due to the illness, he could not make it. It was a great regret that I could not have the honor of meeting him. The following paper was dedicated to his memory.

W.Y.C. Chen and A.M. Fu, Context-free grammars for permutations and increasing trees, Adv. in Appl. Math. 82 (2017) 58-82.

D. Dumont

D. Foata was in contact with Dumont regarding his plan to visit China, and Professor Foata was so kind as to inform me that Professor Dumont could no longer travel or possibly send out a message. An account of Dumont's work on grammars was given by Foata:

D. Foata, Dominique Dumont (1947 - 2016), Sém. Lothar. Combin. 76 ([2016-2017]) Art. B76z.

A grammatical calculus for the Eulerian polynomials

For the grammar G, we have

$$D(y^{-1}) = -xy^{-1}.$$

The following observation in Chen and Fu (2017) is instrumental for deriving relations of the Eulerian polynomials.

For $n \geq 0$,

$$D^n(xy^{-1}) = xy^{-1}(y-x)^n. \qquad (*)$$

For $n = 1$, we have

$$D(x^{-1}y) = x^{-1}y(x-y). \qquad (**)$$

Since $D(x-y) = 0$, we get (*).

A grammatical calculus for the Eulerian polynomials

The following generating function of $A_n(x)$ is well-known.

Theorem

$$\sum_{n\geq 0} A_n(x)\frac{t^n}{n!} = \frac{1-x}{1-xe^{(1-x)t}}.$$

Let us demonstrate that the above relation can be deduced by employing the grammatical calculus.

Grammatical calculus for the Eulerian polynomials

For a Laurent polynomial f in x and y, define

$$\mathrm{Gen}(f,t) = \sum_{n\geq 0} D^n(f)\frac{t^n}{n!}.$$

If g is also a Laurent polynomial in x and y, then we have the multiplicative property (or the homomorphism)

$$\mathrm{Gen}(fg,t) = \mathrm{Gen}(f,t)\,\mathrm{Gen}(g,t). \tag{*}$$

Noting that $A_n(x) = D^n(y)|_{y=1}$ for $n \geq 0$, we are prompted to compute the generating function $\mathrm{Gen}(y,t)$. It follows from (*) that

$$\mathrm{Gen}(y,t) = \frac{1}{\mathrm{Gen}(y^{-1},t)}.$$

A grammatical calculus for the Eulerian polynomials

Theorem (Chen-Fu, 2017)

We have

$$\mathrm{Gen}(y^{-1},t) = \frac{1 - xy^{-1}e^{(y-x)t}}{y-x}.$$

Proof. Since $D(y^{-1}) = -xy^{-1}$, we have

$$\mathrm{Gen}(y^{-1},t) = \sum_{n\geq 0} D^n(y^{-1})\frac{t^n}{n!} = y^{-1} - \sum_{n\geq 1} D^{n-1}(xy^{-1})\frac{t^n}{n!}.$$

Recalling that $D^{n-1}(xy^{-1}) = xy^{-1}(y-x)^{n-1}$ for $n \geq 1$, we get

$$\begin{aligned}
\mathrm{Gen}(y^{-1}) &= y^{-1} - \sum_{n\geq 1} xy^{-1}(y-x)^{n-1}\frac{t^n}{n!} \\
&= y^{-1} - \frac{xy^{-1}}{y-x}\left(e^{(y-x)t} - 1\right)
\end{aligned}$$

A grammatical calculus for the Eulerian polynomials

Therefore,

$$\mathrm{Gen}(y,t) = \frac{y-x}{1-xy^{-1}e^{(y-x)t}}.$$

Setting $y=1$, we are led to the generating function of $A_n(x)$.

It should be noted that the bivariate version of the Eulerian polynomials was also investigated by Carlitz and ***.

Exterior peak of a permutation

Given a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ on $[n]$, an index is called an *exterior peak* if $\pi_1 > \pi_2$ for $i = 1$ or $\pi_{i-1} < \pi_i > \pi_{i+1}$ for $i < i < n$. Let $T(n,k)$ be the number of permutations on $[n]$ with k exterior peaks and let

$$T_n(x) = \sum_{k\geq 0} T(n,k)x^k.$$

Theorem (Gessel)

We have

$$\sum_{n=0}^{\infty} \frac{T_n(x)t^n}{n!} = \frac{\sqrt{1-x}}{\sqrt{1-x}\cosh(t\sqrt{1-x}) - \sinh(t\sqrt{1-x})}.$$

Proper double descents of a permutation

The number of proper double descents of a permutation has been extensively studied. An index i of $\pi = \pi_1\pi_2\cdots\pi_n$ on $[n]$ is called a *proper double descent* if $3 \le i \le n$ and $\pi_{i-2} > \pi_{i-1} > \pi$. Denote by $U(n,k)$ the number of permutations on $[n]$ with k proper double descents and let

$$U_n(y) = \sum_{k\ge 0} U(n,k)y^k.$$

Theorem (Elizalde and Noy, Barry, Basset)

We have

$$\sum_{n=0}^{\infty} U(n,0)\frac{t^n}{n!} = \frac{\sqrt{3}}{2}\frac{e^{t/2}}{\cos(\sqrt{3}t/2)+\pi/6}.$$

Joint distribution

To consider the joint distribution of the number of exterior peaks and the number of proper double descents over permutations, define $P_n(i,j)$ to be the number of permutations on $[n]$ with i exterior peaks and j proper double descents. Let

$$P_n(x,y) = \sum_{i,j} P_n(i,j) x^i y^j$$

where $0 \leq j \leq n-1$ and $2i + j \leq n$.

Joint distribution

Fu (2018) found a context free grammar and a grammatical labeling of permutations to generate the polynomials $P_n(x, y)$. Define the polynomials $P_n(x, y, z, w)$ in four variables as

$$P_n(x, y, z, w) = \sum_{i,j} P_n(i, j) x^i y^j z^{i+1} w^{n-2i-j},$$

where i and j are in the same range as before. Let G be the grammar

$$G = \{x \to xy, \quad y \to xz, \quad z \to zw, \quad w \to xz\},$$

and let D be the formal derivative with respect to G.

Fu (2018) proved that for $n \geq 0$,

$$D^n(z) = P_n(x, y, z, w),$$

and obtained the generating function for the joint distribution, which implies the results of Gessel, Elizalde-Noy, Barry and Basset.

Theorem (Fu-2018)

$$\sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{n!} =$$

$$\frac{2\sqrt{(1+y)^2-4x}\,e^{t/2\cdot(1-y+\sqrt{(1+y)^2-4x})}}{1+y+\sqrt{(1+y)^2-4x}-(1+y-\sqrt{(1+y)^2-4x})e^{t\sqrt{(1+y)^2-4x}}}.$$

A.M. Fu, A context-free grammar for peaks and double descents of permutations, Adv. in Appl. Math. 100 (2018), 179-196.

III. The γ-Expansion of the Eulerian Polynomials

The γ-expansion of the Eulerian Polynomials

The following theorem is concerned with the celebrated gamma positivity of the Eulerian polynomials.

Theorem (Foata and Schützenberger)

For $n \geq 1$, $A_n(x)$ can be written as

$$A_n(x) = \sum_{k=1}^{[(n+1)/2]} \gamma_{n,\,k}\, x^k (1+x)^{n-2k+1}$$

with nonnegative coefficients $\gamma_{n,k}$.

The above expression is called the γ-expansion of $A_n(x)$.

The γ-expansion of the Eulerian Polynomials

Remarkably, Foata and Schützenberger found a combinatorial interpretation of the coefficients $\gamma_{n,k}$.

Theorem (Foata and Schützenberger)

For $n \geq 1$ and $1 \leq k \leq [(n+1)/2]$, $\gamma_{n,k}$ equals the number of permutations of $[n]$ with k descents, but no double descents.

By a change of the grammar G, the positivity of the γ-expansion becomes transparent. This beautiful observation is due to Ma, Ma and Yeh.

Let's see how it works. Since

$$\begin{aligned} D(xy) &= (x+y)xy, \\ D(x+y) &= 2xy. \end{aligned}$$

If we treat xy as a variable u and $x+y$ as a variable v, then the of D corresponds to a new grammar

$$H = \{u \to uv, \quad v \to 2u.\}$$

For example, $A_1(x,y) = u$, $A_2(x,y) = D(u) = uv = 2xy(x+y)$, and $A_3(x,y) = D^2(u) = D(uv) = uv^2 + 2u^2 = xy(x+y)^2 + 2x^2y^2$.

The γ-expansion of the Eulerian Polynomials

We shall demonstrate that the grammar H is vital to provide a combinatorial explanation of the numbers $\gamma_{n,k}$, which turns out to be in one-to-one correspondence with the original interpretation of Foata and Schützenberger.

The γ-expansion of the Eulerian Polynomials

Theorem (Chen-Fu, 2021)

For $n \geq 1$ and $1 \leq k \leq [(n+1)/2]$, the number $\gamma_{n,k}$ equals the number of 0-1-2 increasing plane trees on $\{1, 2, \ldots, n\}$ with k leaves.

The above interpretation is equivalent to a formula given by Han-Ma (2020) in terms of 0-1-2 increasing trees, which in turn is equivalent to a known characterization in terms of increasing binary trees not containing any single left children.

While the above interpretation is merely a restatement of a known fact, it originates from the governing grammar and as will been seen it offers a test ground for the study of the trivariate second-order Eulerian polynomials.

Remark. For the above grammar $G = \{x \to xy,\ y \to xy\}$, the grammatical labeling of a 0-1-2 increasing tree is given as follows. For a 0-1-2 increasing tree T on $[n]$, a leaf is labeled by x, a degree one vertex is labeled by y and a degree two vertex is labeled by 1. A 0-1-2 increasing tree T' on $[n+1]$ can always be obtained from a 0-1-2 increasing tree T on $[n]$ by attaching the vertex $n+1$ to T as a child of a leaf or a degree one vertex.

Furthermore, the grammar H can be utilized to derive the generating function of the polynomials $\gamma_n(x)$, defined by

$$\gamma_n(x) = \sum_{k=1}^{[(n+1)/2]} \gamma_{n,k} x^k.$$

In view of the grammar H, the computation of the polynomials $\gamma_n(x)$ is not much different from that of the André polynomials $E_n(x)$.

The γ expansion of a polynomial whose coefficients are symmetric can be better understood in the framework of symmetric functions in x and y. For the case of two variables x and y, the elementary symmetric functions are $x + y$ and xy.

André polynomials

André polynomials.

IV. The Second Order Eulerian Polynomials

Stirling Permutations. Stirling permutations were introduced by Gessel and Stanley. For $n \geq 1$, let $[n]_2$ denote the multiset $\{1, 1, 2, 2, \ldots, n, n\}$. A permutation $\pi = \pi_1\pi_2 \cdots \pi_{2n}$ of $[n]_2$ is said to be a Stirling permutation if for any $1 \leq j \leq n$ the elements between the two j's in π, if any, are greater than j. A descent and an ascent of π can be defined analogously to the case of ordinary permutations. For a Stirling permutations σ, we adopt the convention that σ is patched with a zero both at the beginning and at the end, that is, $\sigma_0 = \sigma_{2n+1} = 0$.

Some statistics of Stirling permutations

Let Q_n be the set of Stirling permutations on $[n]_2$. For $\sigma \in Q_n$, the sets of ascents, descents and plateaux are defined by

$$\begin{aligned}
\mathrm{Asc}(\sigma) &= \{i \,|\, \sigma_{i-1} < \sigma_i,\ 1 \le i \le 2n\},\\
\mathrm{Des}(\sigma) &= \{i \,|\, \sigma_i > \sigma_{i+1},\ 1 \le i \le 2n\},\\
\mathrm{Plat}(\sigma) &= \{i \,|\, \sigma_i = \sigma_{i+1},\ 1 \le i \le 2n\}.
\end{aligned}$$

We use $\mathrm{asc}(\sigma)$, $\mathrm{des}(\sigma)$ and $\mathrm{plat}(\sigma)$ to denote the cardinalities of $\mathrm{Asc}(\sigma)$, $\mathrm{Des}(\sigma)$ and $\mathrm{Plat}(\sigma)$, respectively.

The equidistribution property

The following equidistribution property is due to Bona.

Theorem (Bona)

For $n \geq 1$, the statistics $\mathrm{asc}(\sigma)$, $\mathrm{des}(\sigma)$ *and* $\mathrm{plat}(\sigma)$ *have the same distribution over* Q_n.

The simplest example is the case $n = 1$, where the Stirling permutation 11 has one ascent, one descent, and one plateau.

The Second-order Eulerian polynomials

The number of Stirling permutations of $[n]_2$ with k descents is called the second order Stirling number, denoted by $C(n,k)$, or $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$.

Gessel and Stanley defined the second order Eulerian polynomials $C_n(x)$ by $C_0(x) = 1$ and for $n \geq 1$,

$$C_n(x) = \sum_{k=1}^{n} C(n,k)x^k.$$

Janson defined the trivariate generating function

$$C_n(x,y,z) = \sum_{\sigma\in Q_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{asc}(\sigma)} z^{\mathrm{plat}(\sigma)}.$$

Janson proved the symmetry property of the joint distribution of the statistics $\mathrm{asc}(\sigma)$, $\mathrm{des}(\sigma)$ and $\mathrm{plat}(\sigma)$ by using an urn model.

Theorem (Janson)

For $n \geq 1$, $C_n(x,y,z)$ is symmetric in x, y, z.

As pointed out by S.-M. Ma, the polynomials $C_n(x,y,z)$ were introduced by Dumont. The symmetry property was also due to Dumont.

A Paper of Dumont in French

In a paper in French published in JCTA, in 1980, Dumont introduced the notion of a repetition of a Stirling permutation, which is called a plateau by Bona. While we do not read French, the machine translation seems to be a good avenue to get around. Although machine translation can play a decisive role at times, it may also have the effects of serious confusions or even terrible misunderstandings. There were some outrageous stories. But in this case, it was great fun.

D. Dumont, Une genéralisation trivariée symétrique des nombres eulériens, J. Combin. Theory, Ser. A, 28 (1980), 307-320.

The e-Positivity of $C_n(x, y, z)$

For $n \geq 1$, since $C_n(x, y, z)$ is symmetric, we can write

$$C_n(x, y, z) = \sum_{i+2j+3k=2n} \gamma_{i,j,k}(x + y + z)^i (xy + xz + yz)^j (xyz)^k.$$

Let $u = x + y + z$, $v = xy + xz + yz$ and $w = xyz$. Then we have

$$\begin{aligned} D(u) &= 3w, \\ D(v) &= 2uw, \\ D(w) &= vw. \end{aligned}$$

It follows that the coefficients $\gamma_{i,j,k}$ are nonnegative.

The e-Positivity of $C_n(x, y, z)$

Using 0-1-2-3 increasing plane trees on $\{1, 2, \ldots, n\}$, Chen-Fu (2021) obtained an interpretation of the coefficients $\gamma_{i,j,k}$.

Theorem (Chen-Fu, 2021)

For $n \geq 1$ and $i + 2j + 3k = 2n + 1$, $\gamma_{i,j,k}$ equals the number of 0-1-2-3 increasing plane trees on $[n]$ with k leaves, j degree one vertices and i degree two vertices.

W.Y.C. Chen and A.M. Fu, A Context-free grammar for the e-positivity of the trivariate second-order Eulerian polynomials, arXiv:2106.08831.

V. Alternatingly Increasing and Spiral Polynomials

Alternatingly increasing and spiral polynomials

A polynomial

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

is called alternatingly increasing if

$$a_0 \le a_n \le a_1 \le a_{n-1} \le \cdots \le a_{\lfloor (n+1)/2 \rfloor}.$$

This is exactly what Bruce Sagan called top interlacing or bottom interlacing.

Alternatingly increasing and spiral polynomials

It is said to be spiral if

$$a_0 \le a_n \le a_1 \le a_{n-1} \le \cdots \le a_{\lfloor (n+1)/2 \rfloor}$$

or

$$a_n \leqslant a_0 \le a_{n-1} \le a_1 \le \cdots \le a_{\lfloor n/2 \rfloor}.$$

If $f(x)$ is spiral and $\deg f = n$, then $x^n f(1/x)$ is alternatingly increasing, and vice versa.

M. Beck, K. Jochemko and E. McCullough, h^*-polynomials of zonotopes, Trans. Amer. Math. Soc., 371 (2019), 2021–2042.

Alternatingly increasing and spiral polynomials

Let $f(x)$ be a polynomial of degree n. There is a unique decomposition, called the symmetric decomposition of the form

$$f(x) = A(x) + xB(x),$$

with the requirement that $A(x)$ and $B(x)$ are symmetric with respect to degrees n and $n-1$, respectively. More precisely, we have

$$A(x) = \frac{f(x) - x^{n+1}f(1/x)}{1-x}, \quad B(x) = \frac{x^n f(1/x) - f(x)}{1-x}. \tag{1}$$

M. Beck and A. Stapledon, On the log-concavity of Hilbert series of Veronese subrings and Ehrhart series, Math. Z., **264** (2010), 195–207.

Proof. By the symmetry conditions on $a(x)$ and $b(x)$, we see that

$$\begin{aligned} x^n A(1/x) &= A(x), \\ x^{n-1} B(1/x) &= B(x). \end{aligned}$$

Since $f(x) = A(x) + xB(x)$, we deduce that

$$x^{n+1} f(1/x) = x^{n+1} A(1/x) + x^n B(1/x) = xA(x) + xB(x),$$

and so

$$f(x) - x^{n+1} f(1/x) = (1-x)A(x).$$

Meanwhile, we obtain that

$$x^n f(1/x) - f(x) = (1-x)B(x).$$

Simplest Examples

$$a + bx = (a + ax) + x(b - a).$$

$$a + bx + cx^2 = (a + (a + b - c)x + ax^2) + x(c - a + (c - a)x),$$

and

$$a + bx + cx^2 + dx^3 = A(x) + xB(x),$$

where

$$\begin{aligned} A(x) &= a + (b + a - d)x + (a + b - d)x^2 + ax^3, \\ B(x) &= (d - a) + (c - a - b + d)x + (d - a)x^2. \end{aligned}$$

A procedure to compute $A(x)$ and $B(x)$.

Intuitively, the coefficients of $A(x)$ and $B(x)$ are determined alternatively in the following order: first, the constant term of $A(x)$, then the highest term of $A(x)$, followed by the highest term of $B(x)$, then the constant term of $B(x)$, then the coefficient of x in $A(x)$, and so forth.

For a polynomial $f(x)$ with alternatively increasing coefficients, it can be written as

$$f(x) = A(x) + xB(x),$$

where both $A(x)$ and $B(x)$ are symmetric and unimodal. This is where the notion of bi-gamma positivity comes into play. Let $m = \lceil \frac{n}{2} \rceil$. Then we have

$$\begin{aligned} A(x) &= \sum_{i=0}^{m} (a_i - a_{n-i+1})(x^i + x^{i+1} + \cdots + x^{n-i}), \\ B(x) &= \sum_{i=0}^{m} (a_{n-i} - a_i)(x^i + x^{i+1} + \cdots + x^{n-i-1}). \end{aligned}$$

The above formulas can be verified directly, or can be derived from the above expressions in (1) for $A(x)$ and $B(x)$.

For example, let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. Assume that

$$a_0 \le a_3 \le a_1 \le a_2.$$

Then

$$A(x) = a_0(1 + x + x^2 + x^3) + (a_1 - a_3)(x + x^2)$$

and

$$B(x) = (a_3 - a_0)(1 + x + x^2) + (a_2 - a_1)x$$

Theorem

Let $f(x)$ be a polynomial of degree n and let

$$f(x) = A(x) + xB(x) \tag{2}$$

be the symmetric decomposition of $f(x)$. Then $f(x)$ is alternatively increasing if and only if $A(x)$ and $B(x)$ have nonnegative coefficients.

We note that the above assertion can also be verified along with the procedure to determine the coefficients of $A(x)$ and $B(x)$.

For example, for the q-derangement number $D_5(q)$, we have

$$D_4(q) = q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6.$$

$* * *$

Question. What is the symmetric decomposition of $D_n(q)$? Study the combinatorial and analytical properties of the decomposition. — Note

Back to the spiral property

This property was first observed by X.D. Zhang in his proof of a conjecture of Chen-Rota on the q-derangement numbers.

$$D_n(q) = [n]! \sum_{k=0}^{n} (-1)^k q^{\binom{n}{2}} \frac{1}{[k]!},$$

where $[n]! = [1][2] \cdots [n]$ and $[n] = 1 + q + \cdots + q^{n-1}$.

While $D_n(q)$ is not symmetric, Chen-Rota observed that the maximum coefficient of $D_n(q)$ appears in the middle. X.D. Zhang noticed the the spiral property, a term I suggested.

The spiral property

He was my postdoc at that time. I was very impressed with his keen observation. While the others might be more impressed by his title as a postdoc, because the title of a postdoc was regarded by some as the highest level of diplomas. Back to that time, there were only a handful postdocs in a province in the south, where one could instantly become renowned for carrying such a title. Indeed, one was not called Dr. so and so, but Postdoc so and so. There was even a special term (Bo-Hou) in Chinese.

$$D_1(q) = 0, \quad D_2(q) = q, \quad D_3(q) = q + q^2,$$

$$D_4(q) = q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6,$$

$$D_5(q) = q + 3q^2 + 5q^3 + 7q^4 + 8q^5 + 8q^6 + 6q^7 + 4q^8 + 2q^9,$$

$$\begin{aligned} D_6(q) &= q + 4q^2 + 9q^3 + 16q^4 + 24q^5 + 32q^6 + 37q^7 + 38q^8 + 35q^9 \\ &\quad + 28q^{10} + 20q^{11} + 12q^{12} + 6q^{13} + 2q^{14} + q^{15}, \end{aligned}$$

$$\begin{aligned} D_7(q) &= q + 5q^2 + 14q^3 + 30q^4 + 54q^5 + 86q^6 + 123q^7 + 160q^8 + 191q^9 \\ &\quad + 210q^{10} + 214q^{11} + 202q^{12} + 176q^{13} + 141q^{14} + 104q^{15} \\ &\quad + 69q^{16} + 41q^{17} + 21q^{18} + 9q^{19} + 3q^{20}. \end{aligned}$$

Look at $D_5(q)$, which is evidently spiral. How could one possibly fail to take a notice of this obvious fact at the beginning? Or it was a destiny not to see it. What a pity!

Remarks

- Although the q-derangement polynomials are not symmetric, in an asymptotic sense, it is almost symmetric. Chen and Wang showed that the limit distribution of the q-derangement polynomials is normal.

W.Y.C. Chen and D.G.L. Wang, The limiting distribution of the q-derangement numbers, European J. Combin. 31 (2010) 2006-2013.

- Chen-Xia noticed that the spiral property follows from a ratio property like the log-concavity, called the ratio monotone property.

The ratio monotonicity

We say that a sequence $a(1), a(2), \ldots, a(n)$ of positive numbers is ratio monotone if

$$\frac{a(1)}{a(n)} \leq \frac{a(2)}{a(n-1)} \leq \cdots \leq \frac{a(\lfloor n/2 \rfloor)}{a(\lceil n/2 \rceil + 1)} \leq 1 \tag{3}$$

and

$$\frac{a(n)}{a(2)} \leq \frac{a(m-1)}{a(3)} \leq \cdots \leq \frac{a(\lfloor n/2 \rfloor + 2)}{a(\lceil n/2 \rceil)} \leq 1, \tag{4}$$

where $\lfloor x \rfloor$ and $\lceil x \rceil$ are the floor function and the ceiling function.

In the case that all the inequalities are strict, we say that the sequence is strictly ratio monotone.

The ratio monotonicity and the spiral property

- Chen and Xia showed that for $n \geq 6$, the q-derangement numbers $D_n(q)$ are strictly ratio monotone, except for the last term when n is even.
- It has also been shown that the Boros-Moll polynomials are ratio monotone.

W.Y.C. Chen and E.X.W. Xia, The ratio monotonicity of the q-derangement numbers, Discrete Math. 311(6) (2011) 393-397

W.Y.C. Chen and E.X.W. Xia, The ratio monotonicity of the Boros-Moll polynomials, Math. Comp. 78 (2009) 2269-2282.

Example

For example, for the case of $D_6(q)$ without the last term, we see that

$$\frac{1}{2} < \frac{4}{6} < \frac{9}{12} < \frac{16}{20} < \frac{24}{28} < \frac{32}{35} < \frac{37}{38} < 1, \tag{5}$$

$$\frac{2}{4} < \frac{6}{9} < \frac{12}{16} < \frac{20}{24} < \frac{28}{32} < \frac{35}{37} < 1. \tag{6}$$

It can be checked that $D_6(q)$ has the following spiral property

$$1 < 2 < 4 < 6 < 9 < 12 < 16 < 20 < 24 < 28 < 32 < 35 < 37 < 38,$$

where the last term is not taken into consideration.

Bi-γ-positive and alternatingly increasing

Theorem (Ma-Ma-Yeh-Yeh, 2021)

Suppose $p(x,y)$ can be expanded as

$$p(x,y)=\sum_{i=0}^{n}y^{i}\sum_{j=0}^{\lfloor (n-i)/2\rfloor}\mu_{n,i,j}x^{j}(1+x)^{n-i-2j}.$$

If $p(x,y)$ is partial γ-positive, $p(x,1)$ is bi-γ-positive and $0\leq y\leq 1$ is a given real number, then $p(x,y)$ is alternatingly increasing.

VI. Grammars and the Ramanujan-Shor polynomials

The Ramanujan-Shor polynomials

For integers $1 \le k \le r+1$, Ramanujan defined the polynomials $\psi_k(r,x)$ by the following relation:

$$\sum_{k=0}^{\infty} \frac{(x+k)^{r+k} e^{-u(x+k)} u^k}{k!} = \sum_{k=1}^{r+1} \frac{\psi_k(r,x)}{(1-u)^{r+k}},$$

and derived the recurrence relation:

$$\psi_k(r+1,x) = (x-1)\psi_k(r,x-1) + \psi_{k-1}(r+1,x) - \psi_{k-1}(r+1,x-1), \quad (7)$$

where $\psi_1(0,x) = 1$, $\psi_0(r,x) = 0$ and $\psi_k(r,x) = 0$ for $k > r+1$.

Berndt, Evans and Wilson obtained another recurrence relation for $1 \leq k \leq r+1$,

$$\psi_k(r,n) = (n-r-k+1)\psi_k(r-1,n) + (r+k-2)\psi_{k-1}(r-1,n).$$

By setting $u = 0$ in (73), Ramanujan deduced the identity for $r \geq 1$,

$$\sum_{k=1}^{r+1} \psi_k(r,x) = x^r.$$

Zeng observed that the polynomials $\psi_k(r,x)$ coincide with the polynomails introduced by Shor as a refinement of Cayley's formula for the numbers of rooted trees on n vertices.

The polynomials $Q_{n,k}(x)$ are called the Ramanujan-Shor polynomials.

Let

$$Q_{n,k}(x) = \psi_{k+1}(n-1, x+n).$$

Then Ramanujan's identity can be recast as

$$\sum_{k=0}^{n-1} Q_{n,k}(x) = (x+n)^{n-1}.$$

It is not hard to give a combinatorial explanation of the recurrence relation (***) derived by Berndt ****. Shor posed the problem of finding a combinatorial interpretation of the recurrence relation (***). While a combinatorial proof has been found by Chen-Guo and a simpler construction has been provided by Guo, none of them seems to be satisfactory.

Chen-Yang (2018) found a grammar for the Ramanujan-Shor polynomials based on the grammar of Dumont, and then gave a grammatical derivation of the recurrence relation (RR).

However, it is still in demand to find a relatively simple bijection for the recurrence relation (RR).

Connection found by Wang and Zhou

Recently, Wang and Zhou showed that the orbifold Euler characteristic of the moduli space of stable curves of genus zero with n marked points turns out to be the Ramanujan-Shor polynomial $Q_{n-1,k+1}(x)$ evaluated at $x = -1$. They discovered the connection by checking with the OEIS. As Stanley mentioned this morning, and he did find something that is already there. For the case of Wang and Zhou, it was beyond their expectation to see their finding. Zhou even got interested in combinatorics in a larger sense, including some work of Rota.

A grammar for Ramanujan-Shor polynomials

Utilizing Shor's recursive procedure to construct rooted trees, Dumont and Ramamonjisoa found a context-free grammar to enumerate rooted trees with a given number of improper edges. They defined a grammar G by the following substitution rules:

$$G = \{A \rightarrow A^3S, \quad S \rightarrow AS^2\}.$$

Dumont-Ramamonjisoa's grammar

Let D denote the formal derivative with respect to G. Dumont and Ramamonjisoa showed that, for $n \geq 1$,

$$D^{n-1}(AS) = A^n S^n \sum_{k=0}^{n-1} b(n,k) A^k,$$

where $b(n,k)$ denotes the number of rooted trees on $[n]$ with k improper edges. Note that $b(n,k) = Q_{n,k}(0)$.

A grammar for Ramanujan-Shor polynomials

Based on the Dumont-Ramamonjisoa grammar, Chen-Yang-Hao (2021) obtain a grammar H to generate the Ramanujan-Shor polynomials $Q_{n,k}(x)$. Let

$$H\colon a \to axy, x \to xyw, y \to y^3w, w \to yw^2,$$

and let D denote the formal derivative with respect to H. For $n \geq 1$, we obtain the following relation

$$D^n(a) = axy^n w^{n-1} \sum_{k=0}^{n-1} Q_{n,k}(xw^{-1}) y^k.$$

A grammar for Ramanujan-Shor polynomials

With the aid of the grammar H, we are led to a simple derivation of the Berndt-Evans-Wilson-Shor recursion.

$$\psi_k(r,n) = (n-r-k+1)\psi_k(r-1,n) + (r+k-2)\psi_{k-1}(r-1,n).$$

It turns out that the grammar H can also be used to derive the Abel identities.

Abel's identities

As will be seen, the Abel identities can be deduced from the Leibniz formula with respect to the grammar H.

Riordan defined the sum

$$A_n(x_1, x_2; p, q) = \sum_{k=0}^{n} \binom{n}{k} (x_1 + k)^{k+p} (x_2 + n - k)^{n-k+q},$$

where $n \geq 1$ and the parameters p, q are integers. He found closed formulas of $A_n(x_1, x_2; p, q)$ for some p and q. These identities were called the Abel identities or the Abel-type identities since the case $(p, q) = (-1, 0)$ corresponds to the classical Abel identity.

Abel's identities

We give a grammar H' based on the grammar H and show that the summations $A_n(x_1, x_2; p, q)$ can be evaluated by using the grammar H'. Using this approach, closed forms can be deduced for $A_n(x_1, x_2; -1, 0)$, $A_n(x_1, x_2; -1, -1)$ and $A_n(x_1, x_2; -2, 0)$ and $A_n(x_1, x_2; -2, -2)$. The case of $A_n(x_1, x_2; -2, -2)$ seems to be new.

The Lacasse identity

There is a grammatical explanation of the identity

$$n^{n+1} = \sum_{k=1}^{n} \sum_{k=0}^{n-j} \binom{n}{j} \binom{n-j}{k} j^j k^k (n-j-k)^{n-j-k}.$$

It was conjectured by Lacasse in the study of the PAC-Bayesian machine learning theory. Since then, several proofs have been found by *******.

VII. Recent Developments

Recent developments

Here only list some recent developments without giving the details. I plan to write a more detailed version of this presentation.

(1) The interpretations of the Eulerian polynomials, the second-order Eulerian polyomials and André polynomials in terms of Young tableaux have been given by Han-Ma (2020).

(2) The flag ascent-plateau polynomial of Stirling permutations is not gamma-positive, and the semi-gamma-positivity of it are proved. (Ma-Ma-Yeh, 2020)

(3) The types A and B alternating Eulerian polynomials have gamma-vectors alternate in sign. (Lin-Ma-Wang-Wang, 2021, Ma-Fang-Mansour-Yeh, 2021)

(4) The enumerative polynomials of Stirling multipermutations by the statistics of plateaux, descents and ascents are partial gamma-positive. (Lin-Ma-Zhang, 2021)

Recent developments

(5) Two connections between fixed point and cycle (p,q)-Eulerian polynomials and multivariate polynomials of colored permutations are established. (Ma-Ma-Yeh-Yeh, 2021)

(6) Chen-Hao-Yang (2021) used grammars to construct stable multivariate polynomials in answer to questions proposed by Haglund and Visontai.

Thank you!