ICECA
International Conference
Enumerative Combinatorics and Applications
University of Haifa – Virtual – September 4-6, 2023

THE IDEA OF A GRAMMATICAL CALCULUS

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A context-free grammar is nothing but a set of production rules or substitution rules as we understand it in computer science. For example, look at the three insanely simple and primitive grammars $G_1 = \{x \to xy,\ y \to xy\}$, $G_2 = \{x \to xy,\ y \to x^2\}$, and $G_3 = \{a \to ax,\ x \to xy,\ y \to x^2\}$, due to Dumont, Chen-Fu and Ma. Nurtured by the Leibniz rule, a grammar begins a new life as a differential operator, not being an annihilation operator, but rather a creation operator. Relying merely on grammars, we could deduce various generating functions and identities without minding the recurrence relations or differential equations. The notion of a constant plays an essential role in this context. In fact, a constant does not have to look like a constant in the usual sense and it is entirely up to the grammar to judge whether something is a constant. Our original aspiration was to seek an alternative symbolic mechanism for the enumeration of combinatorial objects with grammatical labels or patterns without resorting to the magic touch of linear functionals in the umbral calculus. As elusively remarked by Zeilberger, Cayley tried to convince himself of the validity of the symbolic method by means of differential operators and hyperdeterminants. While being ignorant of the early history for whatever reason, we shall instead try and give a clue on how far we could go in this direction, at least on the stage of contemporary enumerative combinatorics. On the ground of the grammatical calculus, one sees the generating function for the Eulerian polynomials, Gessel's formula for the generating function of the left peak polynomials, the formulas of David-Barton, Elizalde-Noy, Stanley, and the identities of Petersen and Stembridge, to name a few. In our picture, it is transparent why it is no coincidence for the hyperbolic functions to crop up in Gessel's formula. Another thing about grammars is that they can be instrumental in devising bijections. For example, the grammar facilitates the construction of a correspondence between permutations and increasing trees that maps certain statistics (several kinds of peaks and runs) of permutations to certain statistics (number of even degree vertices and the number of odd degree vertices) of trees. In doing so, we find a reflection principle that reduces to a grammar assisted bijection between alternating permutations and increasing even trees, which was previously studied by Kuznetsov-Pak-Postnikov. The grammatical approach also applies to the γ-positivity, the stability of multivariate second-order Eulerian polynomials, the Ramanujan-Shor polynomials, and to the orbifold Euler characteristics of the moduli spaces of stable curves as recently investigated by Wang-Zhou. While it is certainly not "Everything, Everywhere, All at Once", perhaps it can be something, somewhere, at times. Thanks to many who have contributed so much to the related topics and we look forward to more entries to add to the list.